

**MATHEMATICAL MODELLING OF HOST-PEST INTERACTION  
IN THE PRESENCE OF INSECTICIDES AND RESISTANCE: A  
CASE OF FALL ARMYWORM**

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## DECLARATION

This project is my original work and has not been presented elsewhere for a degree or any other award.

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## **DEDICATION**

To almighty God for his amazing grace and faithfulness in my life and through the completion of this research project.

To my loving and caring family. Thank you for the constant support, prayers and encouragement you gave me through this research journey. You continually taught me the true virtues to embrace which gave me strength, resilience, and perseverance throughout my research project.

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## **ABBREVIATION AND ACRONYMS**

BT	Bacillus Thuringiensis
CTM	Center Manifold Theorem
FAW	Fall Armyworm
IPM	Integrated Pest Management
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
$R_0$	Basic Reproduction Number
RMS	Resistance Management Strategies
SDGs	Sustainable Development Goals
SI	Susceptible Infected
VIP	Vegetative Insecticidal Proteins

## OPERATIONAL DEFINITION OF TERMS

Equilibrium point	A system is said to be at equilibrium point when all the differential equations are equal to zero.
Endemic pest	A pest population that lasts for a long time in the system.
Epidemiological approach	Approach used to model attributes passed from one individual to another through contact.
Host	An animal or plant on or in which a parasite or pest live and feed.
Interaction	An action that occurs when individuals in a system have a two-way effect on each other.
Normal population	Population in the system without the resistant traits.
Organic maize	Maize population without any insecticidal sprays.
Pest	A destructive insect that attacks agricultural crops reducing their yields.
Resistance	Ability of not being adversely affected by something.
Susceptible population	The population who are free of infection but still at a risk of contracting the infection.

## LIST OF GREEK SYMBOLS

$\alpha$	alpha
$\beta$	beta
$\epsilon$	epsilon
$\delta$	delta
$\gamma$	gamma
$\mu$	mu
$\omega$	omega
$\rho$	rho
$\tau$	tau
$\theta$	theta
$\lambda$	lambda
$\pi$	pi

## ABSTRACT

Several pest management programs have been developed to control the rising agricultural pest populations. However, the challenge of rapid evolution and pest resistance towards the control measures continues to cause high production losses to maize farmers in Africa. Few models have attempted to address the issue of Fall Armyworm (FAW) but have not incorporated the effect of insecticides resistance. The knowledge on the effect of insecticides resistance is still scanty. Models with resistance would help predict the dynamics of FAW population thus mitigate loses. The main objectives of this work were to develop, analyse, and numerically simulate a susceptible- infected deterministic mathematical model expressing the FAW-maize interaction and population dynamics under insecticidal sprays and resistance FAW larvae. Three model steady states are established and their local stability conducted using either the eigenvalue or the Routh-Hurwitz stability criterions and the global stability analyzed using Castillo Chavez, Perron eigen vector, and the Lyapunov methods. An expression for the Basic reproduction number,  $R_0$ , and the sensitivity analysis of its parameter values is provided. Numerical analysis is conducted to various model parameter values. The results established all the model steady states to be locally and globally asymptotically stable at  $R_0 \leq 1$ . Also, resistance  $\omega$  increased the infection rates by increasing the FAW larvae survival rate  $\lambda$  and reducing the insecticidal efficacy  $\delta_R$  and  $\delta_N$ . This work informs the agriculture sector and policy makers on pest control with the best ways to use insecticides to minimize pest resistance and enhance efficacy in production. Pest control measures should be modified to lower the FAW survival rate and all model parameters contributing to resistance formation by FAW larvae in order to minimize FAW- host interaction thus reducing crop damage.

## CHAPTER ONE

### INTRODUCTION

#### 1.1 Background information

Fall Armyworm (FAW), scientifically known as *Spodoptera frugiperda*, is an agricultural pest species of the order lepidoptera, a larval stage for the fall armyworm moth (Alvarenga et al., 2017). It is a polyphagous, sporadic pest that has continuously caused crop destruction and yield losses to both organic and inorganic farmers globally (Rosent rat er et al., 2016). Maize (corn) is considered a staple food and a source of food security to most African countries; however, farmers continue to face the threat of significant production losses due to climate change, pest and diseases (Haftay Gebreyesus Gebreziher, 2020). Recent research studies show that maize is the most preferred host plant of FAW (Westbrook et al., 2016). Various integrated control mechanisms for both organic and inorganic maize farming have been put in place to control the pest-host (FAW-maize) interactions but natural selection and mutation have caused FAW resistance towards the set control mechanisms (Russo et al., 2021).

Research studies show that when a FAW invasion in a maize plantation is improperly managed, it results to a significant reduction in quality and quantity of the harvest (Gichere et al., 2022). In Africa, FAW was first reported in 2016 with the conducive weather conditions and the availability of FAW preferred host favoring rapid FAW reproduction. This makes FAW the most dominant and endemic pest in Africa and thus a great threat to production and food security (Osae et al., 2022). Synthetic insecticides are the main control methods adopted globally against FAW pest invasions especially Africa where governments are spending huge funds buying and distributing insecticides to their farmers (Assefa & Ayalew, 2019). However, continuous use of the insecticides increases the chances of pest resistance against the control method and thus high production cost.

With maize being a staple food in Kenya and also a FAWs most preferred host plants Gichere et al., (2022), the negative impacts of FAW on maize production significantly affects Kenya's big four Agenda of achieving 100% national nutrition and food security

and the entire Africans economic developments (Day et al., 2017). The poor-quality maize yields negatively affect the country's GDP due to poor market access (De Groote et al., 2020). Managing FAW populations is also expensive to most undeveloped African countries. Generally, FAW invasion to the agricultural sector in Africa poses a great threat towards the achievement of the CAADP Malabo declaration of halving poverty by 2025 and the achievements of the 2030 Sustainable Development Goals (SDGs) of improving food security, eradicating poverty, and achieving sustainable production and consumption plans (Osae et al., 2022), (De Groote et al., 2020).

Predicting the dynamics of a pest population and evaluating the existing pest control measures could significantly reduce the number and the cost of pest management thus improving on crop production, food security, and sustainability (Daudi et al., 2022). There is an increased need to study FAW- maize interaction and the application of insecticides with the evolution of resistance to develop better control methods that are more efficient, effective, and economical (De Groote et al., 2020), (Lee et al., 2020). Mathematical modeling offers an avenue to explore such important factors in agricultural production (Lewis et al., 2016). For instance, crop growth models issue physiological approaches for the simulations of pest destructions and crop interactions (Chander & Arya, 2016).

Disease infection rate on the maize population could be decreased through control intervention measures such as chemical insecticides aimed at reducing the susceptible-infected maize contacts rates, (Alemneh et al., 2019). In the present work we will consider chemical insecticides as the major control method against FAW. However, resistance allele and migration rate significantly influences FAW population dynamics (Garcia et al., 2019). Ordinary differential equations (ODEs) have been used in developing and analyzing a stage structured FAW- maize interaction models (Daudi et al., 2022), (Daudi et al., 2021). However, limited attention has been shown to host- pest interaction models particularly in insect pest management measures (Ochwach et al., 2021). Thus, we apply ODEs and the concept of host- pest interaction models to study FAW larvae population dynamics in maize populations.

Various deterministic mathematical models describing the dynamics of agricultural pest population under various pest control measures have been developed (Dhahbi et al., 2020),

(Affi, 2018), (Okhuese & Oduwole, 2020) ,(Chávez et al., 2017). Moreover, mathematical models evaluating the effects of pest and insecticides on crop production have been developed (Daudi et al., 2022), (Misra et al., 2021). However, despite the many pest control models in agricultural production comprehensive parametric research are limited. Through natural selection and mutation, a proportion of FAW pest is considered to express resistance traits against the insecticides thus increasing the host- pest interactions and reducing the insecticides efficacy (Russo et al., 2021) .

To address this gap, this study develops a Susceptible-Infected (SI) compartmental model for two interacting populations; FAW – maize population, assessing the effects of insecticides sprays and resistance factor on the interaction patterns and the population dynamics. SI models are used to model the rate and transmission dynamics of infectious human or plant diseases (Wang, 2022), (Cheneke, Rao, & Edessa, 2022). This study assumes that the FAW larvae depend largely on the maize population for food and survival and considers insecticides which are most commonly used control method against FAW. This research study will increase the understanding of the fall armyworm-maize interaction patterns, and the best control measures to employ when the FAW species is in its larval stage while minimizing the larvae resistance formation. This will significantly improve on crop production enhancing food security and economic development (Sachs, 2012).

## **1.2 Statement of the problem**

Fall Armyworm has annually been reported to cause significant reduction in quality and quantity of harvest in agriculture. Several control measures against Fall Armyworm pest including the (Integrated Pest Management) IPM strategies have been employed but none is fully effective. Despite the global insecticides use in controlling the pest population, concerns over the development of resistance against the insecticides have increased over the recent years (Guo et al., 2020). These increases the need to study FAW- host interaction and the insecticides application with evolution of resistance to develop better control methods that are more effective and economical. Mathematical modeling offers an avenue to explore such important factors in agricultural production sector. However limited mathematical models on FAW- Maize interaction with pesticides and resistance



factor have been developed so far to aid on improving the existing resistance management strategies. Thus, our study will develop a mathematical model describing FAW larvae interaction with maize growing up to the reproductive stage and the effects of insecticides sprays and resistance on FAW interaction with its host plant i.e., maize. This will help better understand the FAW resistance and thus guide farmers and government IPM strategies in better control and management of FAW in maize production.

### **1.3 Research questions**

1. How much does insecticides and resistance factors affect the maize-FAW larvae interactions patterns at any time  $t$ ?
2. What are the results of mathematically analyzing the FAW-Maize interaction model with a comparison between the normal and insecticides resistance larvae?
3. What are the impacts of varying some model parameters to the interactions between the FAW larvae and Maize populations?

### **1.4 Objectives**

The general and specific objectives for the study are provided below.

#### **1.4.1 General objectives**

To develop, analyze, and simulate a mathematical model of host-pest interaction under insecticides and resistance factor. A case study of fall armyworm – maize interaction.

#### **1.4.2 Specific objectives**

The specific objectives of the study are to;

1. Develop a mathematical model of Fall Armyworm-maize interactions with normal and insecticides resistance larvae in two maize sections; with and without insecticides sprays.
2. Conduct mathematical analysis of the model.

3. Perform numerical simulation to determine the impact of insecticides resistance on FAW-host interaction.

### **1.5 Justification of the study**

Maize also called corn, is cultivated throughout the world with the United States, China and Brazil being among the top producing countries (Onu et al., 2019). About one-third of the world's population depend on maize as a daily staple food. Maize is popular due to its multiple functionalities as a source of food for both human and animals (Kurt et al., 2016). It is also used in industrial production of oil, alcohol, ethanol etc. Thus, maize is considered as an important crop economically in most parts of the world including Africa where it is a staple food. However, the crop continues to face the problem of pests and diseases which significantly lowers the quality and quantity of harvest. An example in Kenya, attack by stem borers an example FAW larvae and other insect's pests is considered the main shortcoming on maize production. Despite the global pesticides use in controlling the pest population, concerns over the development of resistance against the pesticides have increased over the recent years (Guo et al., 2020).

No single pest resistant management strategy can meet all the requirements for managing FAW resistance, thus there is increased importance in analyzing and evaluating the factors under its management scenarios (Alphey & Bonsall, 2018). Mathematical modeling and computer simulations have been used to analyse resistance evolution and to evaluate potential resistance management strategies (Stratonovitch et al., 2014a). This will improve on maize production enhancing food security in line with the Sustainable Development Goals (SDGs) of improving food security, eradicating poverty and achieving sustainable production and consumption plans (Assembly, G. 2015).

### **1.6 Scope of the study**

The study develops an epidemiological model on fall armyworm (*Spodoptera frugiperda*) interaction with maize under insecticides and resistance factor. We consider maize population growing at any time  $t$  interacting with fall armyworm larvae expressing normal and resistance traits. Although many control measures against FAW exist we only

consider insecticides control method. The FAW population is assumed to reproduce very fast due to its short life cycle while the plant population remains constant. We use secondary data for the variables and parameters from literature to validate the model.

### **1.7 Significance of the study**

In this study, we review the insecticides spray as a control measure of Fall Armyworm larvae with some FAW larvae expressing the resistance trait against the insecticides. We develop a deterministic model of FAW larvae interacting with maize growing to reproduction and the impacts of insecticides and resistance factors to interaction patterns. This study will be very useful in improving the agricultural Resistance Management Strategies (RMS) and Integrated Pest Management (IPM) strategies. It will be applied in managing FAW population in maize fields thereby increasing on maize yields quantity and quality. This will further improve on food security in line with the Sustainable Development Goals (SDGs) of improving food security, eradicating poverty and achieving sustainable production and consumption plans (Assembly, G. 2015).

## CHAPTER TWO

### REVIEW OF LITERATURE

#### 2.1 The Fall armyworm

According to a study conducted by Assefa and Ayalew, (2019), the length of FAW life cycle depends on the season. It is observed to take between 30 days in summer to 90 days in winter. The FAW egg stage take a duration of two to three days during summer season. The female FAW lay eggs in large masses of about 100 to 200 and has an ability to lay up to 1500 eggs in a lifetime. The eggs hatch into the larvae stage which is divided into six instars. The larval stage takes between 14 days during summer to 30 days during winter Chhetri, (2019). The larvae stage interacts with the maize plants by consuming the leaves, ribs, and the stalks leaving the plant with a torn appearance. Larvae also attack the plant by destroying the bud, whorl, and the ear. The pupa stage for the FAW takes place in the soil which later develops into an adult FAW. FAW is described as a highly destructive agricultural pest with high reproduction rate and a short life span. However, limited solutions towards sustainable FAW management in Africa and Asia exists till date. Thus, increased need for a science based, inclusive and well balanced IPM strategies (B. M. Prasanna, 2017).

A study by Day et al. (2017), reviewed a variety of research works on the introduction, distribution and management of FAW in Africa. FAW considered native to the tropics of America has in the recent years invaded Africa causing significant maize and crop damage of approximated 25% to 70% loss. Various control methods have been adopted towards large scale eradication of FAW including use of chemical and cultural control measures with minimal success. In near future gathering and analyzing experience would help in designing and testing a suitable control method for the FAW pest.

A research study conducted by B. Prasanna et al. (2018), shows that it is very essential to understand the FAW biology before taking any control measure. This helps improve on scientific investigations and achieve immediate solution. FAW has been observed to possess characteristic factors different from other agricultural pests commonly dealt with

before. Some of them being its polyphagous nature, its high spreading and migratory trait and its ability to survive throughout all seasons.

Abdirahman Yonis (2019), conducted a study evaluating the impacts of FAW on maize production, its economic impact on the yields, and the control measures of FAW in Afoi district. The results of the studies showed that 89% of their interviewers agreed to the negative impact of FAW on maize production and over 90 percent agreed to FAW reducing quality and quantity of harvested yield. FAW was reported to be a destructive pest causing significant food damage and need to be controlled effectively.

Several fields evolved resistant fall armyworm populations have been recorded globally, they include those showing resistance to variety of chemical pesticides and genetically modified Bt crops, (Chandrasena et al., 2018); (Gutiérrez-Moreno et al., 2019). The study represents a valuable advance toward improving management strategies for fall armyworm.

According to Jin et al. (2018), crops genetically engineered to produce insecticidal proteins from the bacterium *Bacillus thuringiensis* (Bt) kill some major pests and reduce use of insecticide sprays. However, evolution of pest resistance to Bt proteins decreases these benefits. Better understanding of the genetic basis of resistance to Bt crops is urgently needed to address this problem. Clarifying the development of pesticide and Bt resistance in fall armyworm would be very beneficial in providing scientific support in the acceptance and use of Bt biopesticides. The FAW is known for building-up resistance to insecticides very quickly.

Morales et al. (2021), evaluated different aspects of resistance of cultivators cropped by farmers in Kenya to FAW larvae feeding under laboratory and field conditions. Feeding of FAW neonate larvae in no choice and choice experiments, development of larvae/pupae food assimilation under laboratory condition and plant damage in a field experiment was conducted. Results showed that there are differences between cultivators but high levels of resistance to larvae feeding was not found.

## **2.2 Mathematical models of FAW-host interaction, impacts on production and resistance factor.**

According to Muthuri (2009), scientist have made positive progress through modelling which has helped in new scientific advancements and disease conquest thus improving the human living standards today. But despite various success achieved through modelling, scientists still find new research gaps the models fail to explain or correctly predict. Modelling is an important research tool ever developed to help solve daily challenges we face. Thus, it is important to revise and improve existing models as new information gets discovered.

Luo et al. (2011), conducted a study modelling mutation and selection using basic mathematics with evolutionary modelling. According to the study evolution of resistance has caused significant scientific problems. This has resulted to increased research in evolutionary modelling. This research study introduced mathematical methods used to create dynamic models of evolution. They reviewed classical application of evolutionary modelling including methods to suppress evolution and used selection to optimize treatment. Dynamic modeling of evolutionary processes is an emerging field with important medical applications.

Understanding how different farming practices and environmental factors interact with a pest population resulting to the spread of resistance is quite a challenging experiment at realistic spatial and temporal scales. Mathematical modelling and computer simulation have, therefore, been used to analyse resistance evolution and to evaluate potential resistance management tactic. The study used an individual based model to simulate the effects of farming practices on pest population dynamics and the impacts different control strategies have on resistance development rate. However, the model failed to incorporate a host-pest interaction in its control strategies and thus compare the resistance development rate (Stratonovitch et al., 2014b).

Anggreini (2016), conducted a study examining the stability of differential equations of the mathematical model on host-parasitoids interactions. The research involved the application of mathematical models in the field of biology that examined the interaction

of the two populations, i.e., host populations and parasitoid populations. In the study the parasitoid population slowly kills the host population by living aboard and by taking food from the host population it occupies. Differential equations were used to construct a mathematical model particularly focused on the stability of the local mathematical model of interaction of two differential equations that is; host and parasitoid populations. The stability in the study was stable equilibrium points from the characteristic equations of host-parasitoid interactions. The study further determines the eigen values of the Jacobian matrix to indicate whether the stable equilibrium points was asymptotically stable or not.

Slater et al. (2017), developed a model evaluating and extending advices towards pest resistance management and in understanding needs and opportunities offered by new control techniques. The research study observed and simulated levels of efficacy obtained with lambda-cyhalothrin(L) and pymetrozin (p) sprays as a function of time since application. According to the study, no single management strategy fully helps in managing resistance factor in pests. It's of great importance to evaluate factors prevailing in adopted pest management strategies. However, the model had no pest- host interaction in its resistance management plan. Effectiveness of a mixture strategy declined with reduction in grower compliance. At least 50% compliance was needed to cause some delay in host resistance development.

Although pesticides are considered as the major cause of evolution of insect pests, the evolutionary process that give rise to insecticides resistance remain poorly understood. Insecticide resistance is reported to increase with increase in insecticide use. The research study used 532 records of the arthropod pesticide resistance database covering 20 species and applied survival analysis to model the number of generations from insecticide use to first report of resistance. Arthropod species significantly varied in how rapidly they evolved resistance to new insecticides regardless of their chemistry. This study laid bases to understanding how insecticides resistance evolves, guiding on future management strategies and further resistance management research studies (Brevik et al., 2018).

Garcia et al. (2019), developed a computational model to describe the dynamics of FAW by considering the crop dynamics, temperature changes and population genetics. The results were compared with insect monitoring data and well fitted in relation to population

dynamics. The model estimated the parameters associate with insect dynamics including, resistance-allele frequency (0.15), migration rate (0.48) and rate of larval movement (0.04). A posterior sensitivity analysis indicated that the frequency of the resistance allele most influenced the model, followed by the migration rate. The study model constituted important tools that could be used in designing IPM programs to manage FAW. It could also be applied in predicting the population dynamics of FAW in affected areas. This study however did not consider the insecticides-resistance factor in managing FAW population dynamics.

Daudi et al. (2021), explored the dynamics and implication of FAW outbreak. The study proposed a new dynamical system for maize biomass and FAW interaction via Caputo fractional-order operator. A generic model with two sub models to assess the effect of FAW infestation on maize at the vegetative and reproductive stages was developed. Numerically the two sub models under two different cases were analyzed. The first case involved different numbers of adult FAW in the field at  $t = 0$  and the second case concerned the existence of factors that led to immigration of adult FAW at time (t). Finally, a simulation of the Caputo system using the Adam-Bashforth- Moulton method was conducted. The work was not exhaustive and future expectations for deploying optimal control of FAW to maximize yield and improve on quality and quantity of maize production was advised.

A study by Daudi et al. (2022), presented a non-autonomous model with a Hollings type II functional response to study dynamics for FAW- maize interaction in a periodic environment. The model was analyzed by investigating positive invariance, boundedness, permanence, global stability and non-persistence. The model was then extended to cover time dependent controls. The study investigated the impact of reducing FAW egg and larvae population through traditional and chemical insecticides. Seasonal variation plays a significant role on all FAW stages. The modeling approach presented here provided a framework for designing effective control strategies to manage the fall armyworm during outbreaks. However, resistance factor was not discussed in the model.



### **2.3 Research gap**

Various deterministic mathematical models describing the dynamics of agricultural pest population under various pest control measures have been developed. Moreover, mathematical models evaluating the effects of pest and insecticides on crop production have also been developed. However, limited attention has been shown to host- pest interaction models particularly in insect pest management measures (Ochwach et al., 2021). Thus, we apply ODEs and the concept of host- pest interaction models to study the interaction of FAW larvae population dynamics in maize populations. We also incorporate the effects of insecticides resistance by the FAW larvae on the host- pest interaction dynamics and on the maize population.

## CHAPTER THREE

### MATERIALS AND METHODS

#### 3.0 Introduction

In this chapter, we describe the methodology used in this research work which involves developing, analyzing and numerically simulating an epidemiological Fall armyworm-maize interaction model. We present the relationship between variables and model parameters using a system of autonomous nonlinear ordinary differential equations. We develop a deterministic epidemiological model of FAW-host interaction with six compartmental classes assuming the dynamics of FAW population approaching an equilibrium or steady state and the maize population at the two sections being constant. We analyze the model by calculating the basic reproduction number ( $R_0$ ) using the next generation matrix, carrying out both local and global stability analysis of the equilibria points using the eigenvalue and Lyapunov stability analysis theory and determine the disease-free equilibrium point (DFEP). We test the positivity and boundedness of the model. We then carry out model simulations. Simulations are conducted to analyse the effects of variables and parameter interactions in the model.

#### 3.1 Model Formulation

In this section, we develop and evaluate the dynamics of a Susceptible-Infected (SI) compartmental model for two interacting populations; the maize population and the FAW population. In SI models, infection occurs when a susceptible individual comes into contact with an infected individual thus contracting the disease (Ega, 2022). In this study, maize population transits from the susceptible into infected compartmental class after making contact with either the resistant or the normal larvae which transmits the disease. The FAW larvae population has been divided into two compartmental classes, that is; the normal larvae  $L_N(t)$  and the resistant larval population  $L_R(t)$ . A proportion of normal larvae progresses into resistance larvae at a constant rate  $\omega$  after contact with the insecticides. The natural recruitment rate contributing to the FAW larvae population is a constant rate  $e_N$  through a survival rate  $\lambda$ .

There are two maize sections; the organic (O) and the insecticidal maize sections (I). The organic section (O) is without any FAW control methods while the insecticidal section (I) is under insecticidal spraying. The natural recruitment rates of the maize population occur at a constant rate  $\rho$ . The two maize sections interact naturally with both the normal larvae  $L_N(t)$  and the resistance larvae  $L_R(t)$ . The FAW larvae  $N_L(t)$  infects the maize population  $N_M(t)$  through contact. The maize population in the two sections (O and I) then progress from susceptible  $S^M(t)$  into the infected  $I^M(t)$  compartments under two forces of infection  $\beta_O$  and  $\beta_I$  respectively. Susceptible and infected maize populations at any time t in organic section (O) are denoted as  $S_O^M(t)$  and  $I_O^M(t)$  respectively, while the susceptible and infected maize population at any time t in insecticidal section (I) are denoted as  $S_I^M(t)$  and  $I_I^M(t)$  respectively.

FAW population  $N_L(t)$  slowly kills the host maize population  $N_M(t)$  by residing in it, infecting and feeding on the maize biomass (Daudi et al., 2022). This contributes to an increased recruitment rate of FAW at a rate  $(\pi_1, \pi_2)$  and a reduction in maize population over time. The natural harvesting rate of the maize population is a constant rate denoted by  $\pi$ . The FAW larvae population declines at a constant rate  $\mu_L = \mu_1 + \mu_2$  which is either by the natural death rate  $\mu_1$  or progression into the pupal FAW life cycle at a constant rate  $\mu_2$ . The exposure to insecticides causes death of the FAW larvae at constants rates  $\delta$ , with  $\delta_N$  denoting the normal larvae insecticidal induced death rate and  $\delta_R$  denoting the resistance larvae insecticidal induced death rate, with  $\delta_N < \delta_R$ .

The total population in the model at time t will be,

$$N(t) = N_L(t) + N_M(t). \quad (1)$$

Where;  $N_L(t) = L_N(t) + L_R(t)$  and  $N_M(t) = S_O^M(t) + I_O^M(t) + S_I^M(t) + I_I^M(t)$ .

The force of infection at any time t in organic section (O) is denoted as  $\beta_O$  while in the insecticidal section (I) it is denoted as  $\beta_I$ ,

$$\beta_O = \eta_O \left( \frac{L_N + \epsilon L_R}{N_M} \right), \quad \beta_I = \eta_I \left( \frac{L_N + \epsilon L_R}{N_M} \right) \quad (2)$$

Where  $\beta_0 > \beta_I$  and  $0 < \beta_0, \beta_I < 1$ ,  $0 < \varepsilon < 1$  and infection coefficients  $\eta_0 > \eta_1$ .

### 3.2 Model assumptions

In this study, the following assumptions were made during model formulation:

- a) To reduce model complexity, the model only considers one larvae stage of the FAW life cycle. The stages are represented by the FAW recruitment rate  $e_N$  and progression rate  $\mu_2$ .
- b) FAW larvae  $N_L(t)$  is the only pest interacting with the maize population at time  $t$ .
- c) Insecticidal sprays are the only control methods adopted against the FAW population.
- d) The term normal larvae denote the larvae not expressing the resistance traits.
- e) Negligible immigration and emigration rates of the adult FAW larvae population.
- f) Homogenous mixing of the FAW and maize population at any time  $t$ .

### 3.3 Model variables and parameters

The following state variables and parameters are discussed as used in the model formulation.

**Table 1:** Description of the model state variables

State variable	Description
$S_O^M(t)$	Susceptible maize population in the organic section at any time t.
$I_O^M(t)$	Infected maize population in the organic section at any time t.
$S_I^M(t)$	Susceptible maize population in the insecticidal section at any time t.
$I_I^M(t)$	Infected maize population in the insecticidal section at any time t.
$L_N(t)$	Normal larvae population at any time t.
$L_R(t)$	Resistant larvae population at any time t.

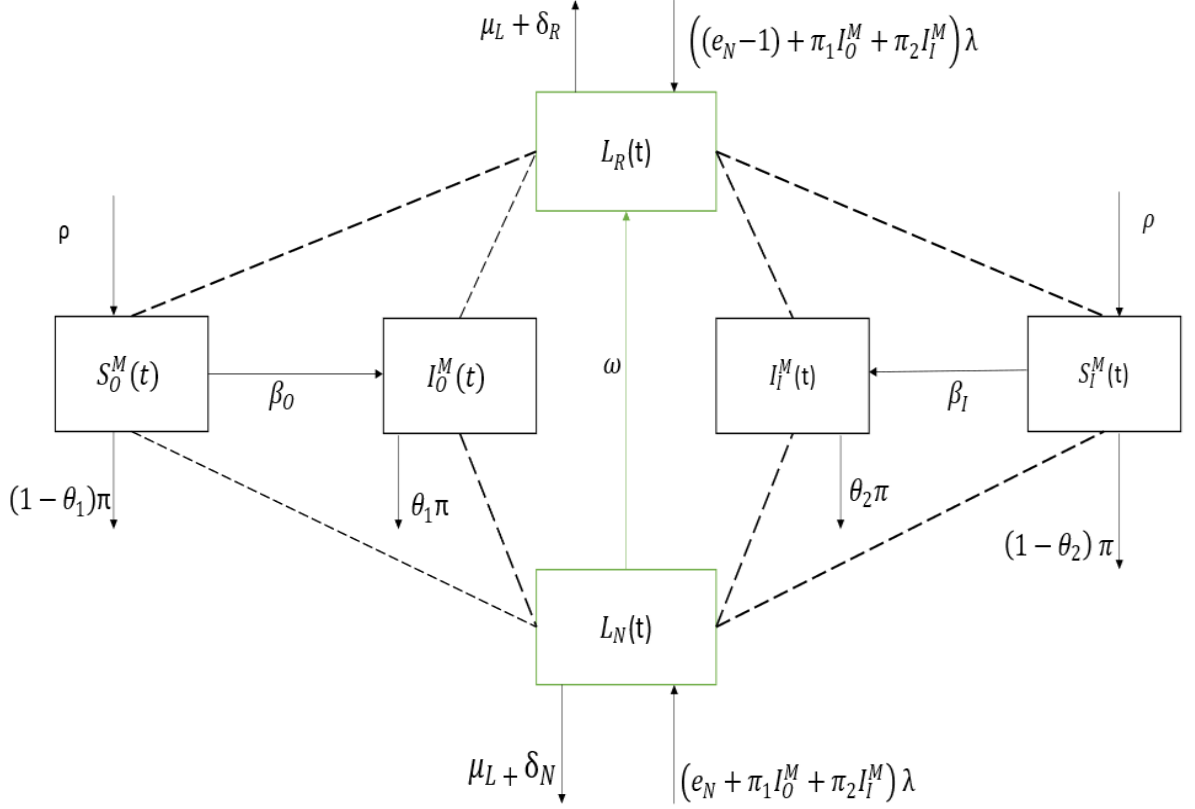
**Table 2:** Description of Model parameters

Model Parameter	Description
$\beta$	The force of infection from susceptible to infected maize population
$\theta_1$	The harvesting rate of organic maize population $N_O^M(t)$
$\theta_2$	The harvesting rate of insecticidal sprayed maize population $N_I^M(t)$
$\eta_0$	The infection coefficient in $\beta_0$
$\eta_1$	The infection coefficient in $\beta_1$

$e_N$	The natural recruitment rate of $N_L(t)$ from the naturally occurring FAW population.
$\pi$	The lost maize biomass in $N_M(t)$ at any time t due to caterpillar attack.
$\omega$	The rate at which normal larvae progress into the resistance larvae population.
$\rho$	The natural recruitment rate of maize biomass into the maize population.
$\mu_L$	The total population decrease rate of $N_L(t)$ at any time t.
$\mu_2$	The rate of larvae progression to the pupal FAW life cycle.
$\mu_1$	The death rate of the FAW larvae at any time t due to natural causes.
$\delta$	The insecticidal-induced death rate in the $N_L(t)$ population
$\lambda$	The survival rate of $L_N(t)$ and $L_R(t)$ from the egg stage of the FAW population at any time t
$\pi_1, \pi_2$	The maize biomass from the $I_O^M(t)$ and $I_I^M(t)$ classes respectively, contributing directly to the $N_L(t)$ classes increased the natural recruitment rate.

### 3.4 Model flow chart

Considering the state variables, parameters, and assumptions discussed above, we developed the model flow chart below,



**Figure 1:** Flow diagram for the FAW larvae - maize interaction model.

### 3.5 Model equations

From the assumptions and the model flow diagram above, the following system of equations are derived. The model is described by the following system of ordinary differential equations:

$$\frac{dS_o^M}{dt} = \rho - \beta_o S_o^M - (1 - \theta_1) \pi S_o^M, \quad (3)$$

$$\frac{dI_o^M}{dt} = \beta_o S_o^M - \theta_1 \pi I_o^M, \quad (4)$$

$$\frac{dS_i^M}{dt} = \rho - (1 - \theta_2) \pi S_i^M - \beta_i S_i^M, \quad (5)$$

$$\frac{dI_i^M}{dt} = \beta_i S_i^M - \theta_2 \pi I_i^M, \quad (6)$$

$$\frac{dL_N}{dt} = \lambda(e_N + \pi_1 I_O^M + \pi_2 I_I^M) - (\omega + \mu_L + \delta_N)L_N, \quad (7)$$

$$\frac{dL_R}{dt} = \left( (1 - e_N) + \pi_1 I_O^M + \pi_2 I_I^M \right) \lambda + \omega L_N - (\mu_L + \delta_R)L_R. \quad (8)$$

### 3.6 Model Analysis

In this section we will investigate, the boundedness and the positivity of the model, determining the equilibrium points, the computation of the basic reproduction number and its sensitivity analysis, and finally the stability analysis of the equilibrium points.

#### 3.6.1 Positivity of solutions and invariant region

Positivity ensures that the model is well-posed and the equations lie on the feasible region of the system thus realistic in representing pest-host interaction with positive values (Iqbal et al., 2020). Since the model system describes a living population of FAW larvae- maize interaction, then the state variables and the model parameters are positive at any time  $t > 0$ . Generally, the solution to the initial value problems defined in  $\Omega$  exist and are unique in the given interval. They remain bounded in the positively invariant and attracting region  $\Omega$ ; hence the model is defined to be biologically and epidemiologically well posed and the dynamics of the model can be sufficiently studied in  $\Omega$ .

#### 3.6.2 Equilibrium points analysis

These are the points when the system is in a stationary point or a steady state. They are also known as fixed points or critical points or singularity point.

There are various types of equilibrium points in epidemiological modelling. In this study we discuss the Disease-free equilibrium point ( $E_0$ ), Insecticidal free equilibrium point ( $E_C$ ), and Disease endemic equilibrium point ( $E_*$ )

#### 3.6.3 Disease free equilibrium point ( $E_0$ )

This can also be referred to as the pest free equilibrium point. We assume that no FAW larvae population both at resistant and normal stages prevails in the system. Hence the



susceptible maize population in both the two sections i.e., the organic  $S_O^M$  and insecticidal  $S_I^M$  compartmental classes grow exponentially to maturity with no FAW infestation.

The  $E_0$  of the model system of equations (3) – (8) is obtained by setting all the  $\{L_N, L_R, I_O^M, I_I^M\} = 0$ ,  $\beta_0, \beta_1 = 0$ , and  $(S_O^{M'}, I_O^{M'}, S_I^{M'}, I_I^{M'}, L_R', L_N' = 0)$ .

#### **3.6.4 Insecticidal free equilibrium point. ( $E_C$ )**

Insecticides are a control measure used against FAW larvae population in the maize population at the section  $N_I^M(t)$ . At insecticidal free equilibrium point, the FAW larvae is assumed to interact freely with the maize population under no control measures. The FAW larvae both in the normal and the resistant compartment classes coexist naturally with the susceptible maize population in the organic section  $N_O^M(t)$  without any control measures being applied to interfere with the interaction patterns.

Therefore, the insecticidal maize population compartmental classes of the model are set to zero i.e.,  $(S_I^M, I_I^M = 0)$ , With  $\{L_N, L_R, S_O^M, I_O^M\} \neq 0$  and  $(S_O^{M'}, I_O^{M'}, S_I^{M'}, I_I^{M'}, L_R', L_N' = 0)$ .

The insecticidal induced death rates on both the normal and the resistant larvae is also equated to zero. That is  $\delta_N = 0$  and  $\delta_R = 0$ .

#### **3.6.5 Disease endemic equilibrium point ( $E_*$ )**

An endemic equilibrium point is a state in the model system where the disease in the population approaches a constant (Affi, 2018). In the study we let  $E_*$  denote the DEEP.

#### **3.6.6 Basic reproduction number ( $R_0$ )**

$R_0$  is used to estimate the number of secondary infections that could arise when one infected individual is introduced in a completely susceptible population (Guerra et al., 2017). In this study, we use the next generation matrix as discussed in Alemneh et al., (2019) to determine the  $R_0$  of the FAW- maize interaction model. We evaluate the Jacobian matrix of the model system of equations for both the maize and the FAW

compartmental classes at DFEP and determine the spectral radius using Wolfram Mathematica software.

### 3.6.7 Local and global stability analysis

Local stability analysis of the FAW larvae-maize interaction model is useful in investigating whether a system returns to its steady state if a small perturbation is introduced in the system. We develop a Jacobian matrix of the model system of equations (3) – (8) and evaluate its value at: Disease-free equilibrium points, Insecticidal free equilibrium points and at Disease endemic equilibrium points.

The global stability analysis of an equilibrium point helps determine the control conditions for a certain disease. Various methods of determining the global stability of dynamical systems exist today. In this study, we determine the global stability of a system using the, Castillo-Chavez method, Perron Eigen vector method and the Lyapunov method.

### 3.6.8 Sensitivity analysis

We conduct the sensitivity analysis to identify the key parameters that significantly affect the FAW larvae- maize interaction model. We learn whether on increasing a particular parameter value results in an increase in the basic reproduction number. This helps determine the key parameters to consider in the control strategies against the FAW larvae infestation into the maize population by managing the basic reproduction number of the infection.

The normalized forward sensitivity index of a variable,  $R_0$ , that depends differentially on parameter  $P$  is defined by an equation

$$\alpha_{P}^{R_0} = \frac{\partial R_0}{\partial P} \cdot \frac{P}{R_0} \quad (9)$$

Where  $R_0$  represents the Basic reproduction number

$P$  represent all the main parameter.

### 3.7 Numerical simulation

Numerical analysis of the model system of equations (3 – 8) is performed to illustrate the analytic results of the research study. This is achieved by using the parameter values given in Table 5 which are obtained from literature and a few of them estimated. The initial values states are used as  $S_O^M(0) = 1000$ ,  $I_O^M(0) = 0$ ,  $S_I^M(0) = 1000$ ,  $I_I^M(0) = 0$ ,  $L_N(0) = 100$ ,  $L_R(0) = 10$  in addition to the parameter values. The simulations are conducted at a time range of between zero to 60 days. Numerical analysis of the model is conducted using a Matlab inbuilt solver based on Runge-Kutta order 5. The resulted simulation graphs are presented in the figures labelled 2 to 11.

## CHAPTER FOUR

### RESULTS

#### 4.1 Overview

This chapter presents the result findings from the model analysis, conducted using various well explained mathematical methods and numerical simulation conducted using Matlab inbuilt solver with data obtained from literature.

#### 4.2 Positivity and Invariant region

##### 4.2.1 Positivity of solutions

We prove the positivity of solutions of our model system by stating and proving the Theorem1 below, (Cheneke, Rao, & Edesssa, 2022).

##### **Theorem 1**

*Let the initial data set be  $\{S_O^M(0), I_O^M(0), S_I^M(0), I_I^M(0), L_N(0), \text{and } L_R(0) \geq 0\} \in \Omega \in R_+^6$ . Then the solution set  $S_O^M(t), I_O^M(t), S_I^M(t), I_I^M(t), L_N(t), \text{and } L_R(t)$  is positive for all  $t \geq 0$ .*

##### **Proof**

Let the variables  $S_O^M(t), I_O^M(t), S_I^M(t), I_I^M(t), L_N(t), \text{and } L_R(t)$  be solutions to the system of non-negative initial conditions,

$$S_O^M(t) \geq 0, I_O^M(t) \geq 0, S_I^M(t) \geq 0, I_I^M(t) \geq 0, L_N(t) \geq 0, \text{ and } L_R(t) \geq 0. \quad (10)$$

Starting with equation (1),

$$\frac{dS_O^M}{dt} = \rho - \beta_O S_O^M - (1 - \theta_1) \pi S_O^M.$$

Clearly by inspection method,  $\rho \geq 0$  on the assumption of non-negative model variables and parameters.

We need to show that,

$$\frac{dS_O^M}{dt} \geq -\beta_O S_O^M - (1 - \theta_1)\pi S_O^M \quad , \quad \frac{dS_O^M}{dt} \geq -(\beta_O + (1 - \theta_1)\pi)S_O^M .$$

By separation of variables, we obtain

$$\frac{dS_O^M}{S_O^M} \geq -(\beta_O + (1 - \theta_1)\pi)dt.$$

Upon integration with respect to time ( $t$ )

$$S_O^M(t) \geq C_1 e^{-(\beta_O + (1 - \theta_1)\pi)t} \quad \text{where } C_1 \text{ is a constant of integration at } t = 0 .$$

That is,  $C_1 = S_O^M(0)$ .

$$\text{Thus, } S_O^M(t) \geq S_O^M(0)e^{-(\beta_O + (1 - \theta_1)\pi)t} \geq 0. \quad (11)$$

Hence the first equation is positive.

By applying the same procedure, we obtained

$$I_O^M(t) \geq I_O^M(0)e^{-\theta_1\pi t} \geq 0, \quad (12)$$

$$S_I^M(t) \geq S_I^M(0)e^{-((1 - \theta_2)\pi + \beta_I)t} \geq 0, \quad (13)$$

$$I_I^M(t) \geq I_I^M(0)e^{-\theta_2\pi t} \geq 0, \quad (14)$$

$$L_N(t) \geq L_N(0)e^{-(\omega + \mu_L + \delta_N)t} \geq 0, \quad (15)$$

$$L_R(t) \geq L_R(0)e^{-(\mu_L + \delta_R)t} \geq 0 . \quad (16)$$

Hence the solution set  $\{S_O^M(t), I_O^M(t), S_I^M(t), I_I^M(t), L_N(t), \text{ and } L_R(t)\}$  for the model system is proved to be positive at all  $t \geq 0$ . Hence the model equations lie in the feasible region.

### 4.2.2 Invariant region

To prove for boundedness of the solution of our model system, we state and prove the following theorem as applied by (Rodkina & Schurz, 2009).

#### Theorem 2

The solution to the study model equations in section 3.5 is uniformly bounded in a proper subset  $\Omega = \Omega_M \chi \Omega_f$ . Such that:  $\Omega = \Omega_M \cdot \Omega_f = \{S_O^M, I_O^M, S_I^M, I_I^M, L_N, L_R \in R_+^6 | N_M \leq \frac{\tilde{\rho}}{\tilde{\pi}}, N_L \leq \frac{\tilde{\lambda}}{\tilde{\delta}}\}$ .

With  $\Omega_M = \{S_O^M, I_O^M, S_I^M, I_I^M \in R_+^4 | N_M \leq \frac{\tilde{\rho}}{\tilde{\pi}}\}$  and  $\Omega_f = \{L_N, L_R \in R_+^2 | N_L \leq \frac{\tilde{\lambda}}{\tilde{\delta}}\}$ . Then all the solutions to the model system move into and remain in  $\Omega$ .

#### Proof

Getting the boundedness of solution for the maize population at any time t, we take the time derivative of our total maize population along its solution to get,

$$N_M(t) = S_O^M(t) + I_O^M(t) + S_I^M(t) + I_I^M(t),$$

$$\frac{dN_m}{dt} = \rho - (1 - \theta_1)\pi S_O^M - \theta_1\pi I_O^M + \rho - (1 - \theta_2)\pi S_O^M - \theta_2\pi I_I^M,$$

$$\frac{dN_m}{dt} = (\rho + \rho) - \pi(S_O^M + S_I^M) - \pi(\theta_1 S_O^M + \theta_2 S_I^M) - \pi(\theta_1 I_O^M + \theta_2 I_I^M) \leq \tilde{\rho} - \tilde{\pi}N_M \tag{17}$$

$$\Rightarrow \frac{dN_M}{dt} \leq \tilde{\rho} - \tilde{\pi}N_M.$$

Where,  $\tilde{\rho} = (\rho + \rho)$  and  $\tilde{\pi}N_M = \pi(S_O^M + S_I^M) + \pi(\theta_1 S_O^M + \theta_2 S_I^M) + \pi(\theta_1 I_O^M + \theta_2 I_I^M)$ .

Upon integration, through the linear constant coefficient method stated in Lemma 1 (Society & Tables, 2008).

**Lemma:** The linear differential equation  $\frac{dN_M}{dt} \leq \tilde{\rho} - \tilde{\pi}N_M$  where  $a = \tilde{\pi} \neq 0$  and  $b = \tilde{\rho}$  are constants, has infinitely many solutions labeled by  $c \in \mathbb{R}$  as;

$$N_M(t) \leq \frac{\tilde{\rho}}{\tilde{\pi}} + N_M(0)e^{-\tilde{\pi}t}.$$

**proof**

$$N_M'(t) - \tilde{\pi}N_M(t) \leq \tilde{\rho} \text{ also denoted as } N_M' - \tilde{\pi}N_M \leq \tilde{\rho}.$$

Introducing an integrating factor  $\mu$ , we get  $\mu N_M' - \mu \tilde{\pi} N_M \leq \mu \tilde{\rho}$ .

$$\text{Let } -\mu \tilde{\pi} = \mu', \text{ thus, } \mu N_M' + \mu' N_M \leq \mu \tilde{\rho}.$$

The left-hand side can be expressed as a total derivative of a product of two functions,

$$(\mu N_M)' \leq \mu \tilde{\rho}.$$

Replacing the value,  $\mu = e^{\tilde{\pi}t}$  in the equation above to get,

$$(e^{\tilde{\pi}t} N_M)' \leq e^{\tilde{\pi}t} \tilde{\rho}, \quad (e^{\tilde{\pi}t} N_M)' \leq \left(\frac{1}{\tilde{\pi}} e^{\tilde{\pi}t} \tilde{\rho}\right)',$$

$$\left((N_M - \frac{\tilde{\rho}}{\tilde{\pi}}) e^{\tilde{\pi}t}\right)' \leq 0.$$

Upon integration,

$$\left(\left(N_M - \frac{\tilde{\rho}}{\tilde{\pi}}\right) e^{\tilde{\pi}t}\right) \leq N_M(0), \quad N_M(t) \leq \frac{\tilde{\rho}}{\tilde{\pi}} + N_M(0)e^{-\tilde{\pi}t}.$$

$$\lim_{t \rightarrow \infty} N_M(t) \leq \frac{\tilde{\rho}}{\tilde{\pi}}, \text{ thus as } t \rightarrow \infty, \text{ we have } N_M(t) \leq \frac{\tilde{\rho}}{\tilde{\pi}}. \quad (18)$$

Similarly, for the FAW larvae population

$$\begin{aligned} \frac{dN_L}{dt} = & \lambda + \lambda\pi_1 I_O^M + \lambda\pi_2 I_I^M + \lambda\pi_1 I_O^M + \lambda\pi_2 I_I^M - \mu_L L_N - \mu_L L_R - \delta_N L_N - \\ & \delta_R L_R, \end{aligned} \quad (19)$$

$$\frac{dN_L}{dt} \leq \tilde{\lambda} - \tilde{\delta} N_L, \quad (20)$$

Where,  $\tilde{\lambda} = \lambda + \lambda\pi_1 I_O^M + \lambda\pi_2 I_I^M + \lambda\pi_1 I_O^M + \lambda\pi_2 I_I^M$  and  $\tilde{\delta}N_L = \mu_L L_N + \mu_L L_R + \delta_N L_N + \delta_R L_R$ .

Upon integration,

$$\lim_{t \rightarrow \infty} N_L(t) \leq \frac{\tilde{\lambda}}{\tilde{\delta}}, \text{ thus as } t \rightarrow \infty \text{ we have } N_L(t) \leq \frac{\tilde{\lambda}}{\tilde{\delta}}. \quad (21)$$

It then follows that the solution to the model equations exists in the region defined by

$$\Omega = \Omega_M \chi \Omega_f = \{(S_O^M, I_O^M, S_I^M, I_I^M, L_R, L_N) \in R_6^+\}.$$

Such that  $S_O^M \geq 0, I_O^M \geq 0, S_I^M \geq 0, I_I^M \geq 0, L_R \geq 0, L_N \geq 0$ .

$$\text{With } \Omega_M = (S_O^M + I_O^M + S_I^M + I_I^M) \leq \frac{\tilde{\rho}}{\tilde{\pi}} \text{ and } \Omega_f = (L_R + L_N) \leq \frac{\tilde{\lambda}}{\tilde{\delta}}. \quad (22)$$

This proves the boundedness of the solution inside  $\Omega$  which means that all the solutions to the system of the model equations (3) – (8) start in  $\Omega$  and stay in  $\Omega$  at time  $t \geq 0$ . Generally, the solution to initial value problems defined in  $\Omega$  exists and is unique in the given interval. They remain bounded in the positively invariant and bounded region  $\Omega$ ; hence the system is biologically and eco-epidemiologically well-posed and the dynamics of the model can be sufficiently studied in  $\Omega$ .

### 4.3 Equilibrium points analysis

All systems of non-linear differential equations may have none, one, many, or even infinite steady states (Harianto, 2017). In this study, we discuss the disease/larvae free equilibrium points ( $E_0$ ), insecticidal/control free equilibrium points ( $E_C$ ), and disease endemic equilibrium points ( $E_*$ ).

#### 4.3.1 Disease free equilibrium points ( $E_0$ )

The value of  $E_0$  is obtained by setting all the infectious classes to zero,  $\{L_N, L_R, I_O^M, I_I^M\} = 0$  with  $\beta_0, \beta_1 = 0$  and  $(S_O^{M'}, I_O^{M'}, S_I^{M'}, I_I^{M'}, L_R', L_N' = 0)$  to get,



$$S_O^{M0} = \frac{\rho}{(1-\theta_1)\pi}, \quad I_I^{M0} = 0, \quad S_I^{M0} = \frac{\rho}{(1-\theta_2)\pi}, \quad I_I^{M0} = 0, \quad L_N^0 = 0, \quad \text{and} \quad L_R^0 = 0. \quad (23)$$

### 4.3.2 Insecticidal free equilibrium point. ( $E_C$ )

Let  $E_C = (S_O^{Mc}, I_O^{Mc}, S_I^{Mc}, I_I^{Mc}, L_N^c, \text{ and } L_R^c)$  denote the control-free equilibrium points. We solve the control equilibrium points by expressing it in terms of the force of infection  $\beta_0^c$  evaluated at control free equilibrium point. We set  $(S_I^M, I_I^M = 0), \{L_N, L_R, S_O^M, I_O^M\} \neq 0, (S_O^{M'}, I_O^{M'}, S_I^{M'}, I_I^{M'}, L_R', L_N' = 0), \delta_N = 0$  and  $\delta_R = 0$  to get,

$$S_O^{Mc} = \frac{\rho}{(\beta_0^c + (1-\theta_1)\pi)}, I_O^{Mc} = \frac{\beta_0^c}{\theta_1\pi} \left( \frac{\rho}{(\beta_0^c + (1-\theta_1)\pi)} \right), S_I^{Mc} = 0, I_I^{Mc} = 0, L_N^c = \frac{\rho(\beta_0^c\lambda\pi_1 + e_N\pi\lambda\theta_1)}{\pi\psi(\beta_0^c + \pi(1-\theta_1))\theta_1}, L_R^c = \frac{(1-e_N)\lambda}{\mu_L} + \frac{\beta_0^c\lambda\rho\pi_1}{\pi\mu_L(\beta_0^c + \pi(1-\theta_1))\theta_1} + \frac{\rho\omega(\beta_0^c\lambda\pi_1 + e_N\pi\lambda\theta_1)}{\pi\mu_L\psi(\beta_0^c + \pi(1-\theta_1))\theta_1}. \quad (24)$$

Evaluating for the value of  $\beta_0^c$ , as  $\beta_0^c = \frac{1}{2}(b_1 + \sqrt{4b_0 + b_1^2})$ . We let  $\omega = (\omega + \mu_L + \delta_N), \sigma = (\mu_L + \delta_R)$  and  $\psi = (\omega + \mu_L)$ .

$$\text{Where } b_1 = \eta_O\lambda \frac{\rho(\mu_L + \varepsilon(\psi + \omega)\pi_1) + \pi^2\theta_1\varepsilon\psi + e_N\varepsilon\psi - \rho\mu_N\psi\pi\theta_1}{\rho\mu_N\psi}, \text{ and}$$

$$b_0 = \eta_O\lambda \frac{\pi\theta_1\varepsilon\psi + e_N\pi\varepsilon\psi + e_N\rho(\mu_L + \varepsilon\omega + (-1 + e_N)b)}{\rho\mu_N\psi}.$$

### 4.3.3 Disease endemic equilibrium point ( $E_*$ )

An Endemic Equilibrium point is a state in the model system where the disease in the population approaches a constant (Affi, 2018). Let  $E_* = (S_O^{M*}, I_O^{M*}, S_I^{M*}, I_I^{M*}, L_N^*, L_R^*)$  denote disease endemic Equilibrium point.

Solving for the system of equations (3) – (8) in terms of the force of infections  $\beta_0^*$  and  $\beta_1^*$ ,

$$S_O^{M*} = \frac{\rho}{(\beta_0^* + (1-\theta_1)\pi)}, \quad I_O^{M*} = \frac{\beta_0^*}{\theta_1\pi} \left( \frac{\rho}{(\beta_0^* + (1-\theta_1)\pi)} \right), \quad S_I^{M*} = \frac{\rho}{(\beta_1^* + (1-\theta_2)\pi)}, \quad I_I^{M*} = \frac{\beta_1^*}{\theta_2\pi} \left( \frac{\rho}{(\beta_1^* + (1-\theta_2)\pi)} \right),$$

$$L_N^* = \frac{\lambda}{\omega} \left( e_N + \frac{\beta_0^*\rho\pi_1}{\pi(\beta_0^* + \pi(1-\theta_1))\theta_1} + \frac{\beta_1^*\rho\pi_2}{\pi(\beta_1^* + \pi(1-\theta_2))\theta_2} \right),$$

$$L_R^* = \frac{\lambda}{\sigma} \left( 1 - e_N + e_N \frac{\lambda\omega}{\varpi} + \frac{\rho(\varpi+\lambda\omega)\pi_1\beta_0^*}{\pi\omega\theta_1(\pi+\beta_0^*-\pi\theta_1)} + \frac{\rho(\varpi+\lambda\omega)\pi_2\beta_1^*}{\pi\omega\theta_2(\pi+\beta_1^*-\pi\theta_2)} \right). \quad (25)$$

Now substituting the values of  $S_O^{M*}, I_O^{M*}, S_I^{M*}, I_I^{M*}, L_N^*, L_R^*$  in  $\beta_0^*$  and  $\beta_1^*$  and introducing the relation  $\eta_1\beta_0^* = \eta_0\beta_1^*$  which implies  $\beta_1^* = \frac{\eta_1}{\eta_0}\beta_0^*$ ,  $\beta_0^* = \frac{\eta_0}{\eta_1}\beta_1^*$  then replacing into the values of the endemic equilibrium points to obtain the value of  $\beta_1^*$ .

For the model system to lie in the positively invariant region, we let the value of

$$\beta_1^* = -\frac{n_2}{3} + \frac{2^{1/3}(-3n_1+n_2^2)}{3(-27n_0+9n_1n_2-2n_2^3+\sqrt{4(3n_1-n_2^2)^3+(27n_0-9n_1n_2+2n_2^3)^2})^{1/3}} + \frac{(-27n_0+9n_1n_2-2n_2^3+\sqrt{4(3n_1-n_2^2)^3+(27n_0-9n_1n_2+2n_2^3)^2})^{1/3}}{32^{1/3}},$$

$\beta_1^* > 0$  if and only if  $n_0, n_1, n_2 > 0$ .

#### 4.4 Basic Reproduction Number ( $R_0$ )

$R_0$  is used to give an estimate of secondary infections that could arise when one infectious individual gets introduced into a completely susceptible environment (Guerra et al., 2017). We determine the value of  $R_0$  using the next generation matrix as discussed in (Alemneh et al., 2019). We evaluate the Jacobian matrix of the model system of equations at DFEP and determine the spectral radius using Wolfram Mathematica software.

$$\text{DFEP, } \left\{ E_0 = (S_O^{M0}, I_O^{M0}, S_I^{M0}, I_I^{M0}, L_N^0, L_R^0) = \left( \frac{\rho}{(1-\theta_1)\pi}, 0, \frac{\rho}{(1-\theta_2)\pi}, 0, 0, 0 \right) \right\}.$$

The vector for the infected and infectious classes is denoted by  $X = [I_O^M \ I_I^M \ L_N \ L_R]$  and the vector for the uninfected classes is denoted as  $Y = [S_O^M \ S_I^M]$ .

Using the notation  $f$  to denote the matrix for new infection and  $v$  to denote the matrix of the transfer of infections in the system,

$$f = \begin{pmatrix} \beta_O S_O^M \\ \beta_I S_I^M \\ 0 \\ 0 \end{pmatrix}, \text{ and } v = \begin{pmatrix} \theta_1 \pi I_O^M \\ \theta_2 \pi I_I^M \\ -\lambda(e_N + \pi_1 I_O^M + \pi_2 I_I^M) + (\varpi) L_N \\ -((1 - e_N) + \pi_1 I_O^M + \pi_2 I_I^M) \lambda - \omega L_N + (\mu_L + \delta_R) L_R \end{pmatrix}.$$

Evaluating the Jacobian matrices of  $f$  and  $v$  at DFEP, to get  $F = \left[ \frac{\partial f}{\partial X} \right]_{E_0}$ ,  $V = \left[ \frac{\partial v}{\partial X} \right]_{E_0}$ , and  $E_0$  the disease-free equilibrium point.

$$J(f)_{E_0} = F = \begin{pmatrix} \frac{\partial(\beta_O S_O^M)}{\partial I_O^M} & \frac{\partial(\beta_O S_O^M)}{\partial I_I^M} & \frac{\partial(\beta_O S_O^M)}{\partial L_N} & \frac{\partial(\beta_O S_O^M)}{\partial L_R} \\ \frac{\partial(\beta_O S_O^M)}{\partial I_O^M} & \frac{\partial(\beta_O S_O^M)}{\partial I_I^M} & \frac{\partial(\beta_I S_I^M)}{\partial L_N} & \frac{\partial(\beta_I S_I^M)}{\partial L_R} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{E_0}. \quad (26)$$

with,  $\beta_O = \eta_O \left( \frac{L_N + \varepsilon L_R}{N_M} \right)$ , and  $\beta_I = \eta_I \left( \frac{L_N + \varepsilon L_R}{N_M} \right)$ .

$$F = \begin{pmatrix} 0 & 0 & \left( \frac{\eta_O}{S_O^{M0} + S_I^{M0}} \right) S_O^{M0} & \left( \frac{\varepsilon \eta_O}{S_O^{M0} + S_I^{M0}} \right) S_I^{M0} \\ 0 & 0 & \left( \frac{\eta_I}{S_O^{M0} + S_I^{M0}} \right) S_I^{M0} & \left( \frac{\varepsilon \eta_I}{S_O^{M0} + S_I^{M0}} \right) S_I^{M0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \tilde{\eta}_O & \varepsilon \tilde{\eta}_O \\ 0 & 0 & \tilde{\eta}_I & \varepsilon \tilde{\eta}_I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (27)$$

With  $S_O^{M0}$ ,  $S_I^{M0}$  being the population of  $S_O^M$ ,  $S_I^M$  is at the disease-free equilibrium point.

We let  $\tilde{\eta}_O = \frac{\eta_O(-1+\theta_2)}{-2+\theta_1+\theta_2}$ ,  $\tilde{\eta}_I = \frac{\eta_I(-1+\theta_1)}{-2+\theta_1+\theta_2}$ ,  $\varpi = (\omega + \mu_L + \delta_N)$  and  $\sigma = (\mu_L + \delta_R)$ .

$$J(v)_{E_0} = V = \begin{pmatrix} \theta_1 \pi & 0 & 0 & 0 \\ 0 & \theta_2 \pi & 0 & 0 \\ -\lambda \pi_1 & -\lambda \pi_2 & (\varpi) & 0 \\ -\lambda \pi_1 & -\lambda \pi_2 & -\omega & (\sigma) \end{pmatrix}. \quad (28)$$

$$V^{-1} = \begin{pmatrix} \frac{1}{\theta_1 \pi} & 0 & 0 & 0 \\ 0 & \frac{1}{\theta_2 \pi} & 0 & 0 \\ \frac{\lambda \pi_1}{\varpi \pi \theta_1} & \frac{\lambda \pi_2}{\varpi \pi \theta_2} & \frac{1}{\varpi} & 0 \\ \frac{\lambda \omega \pi_1 + \lambda \pi_1 \varpi}{\varpi \pi \sigma \theta_1} & \frac{\omega \lambda \pi_2 + \lambda \pi_2 \varpi}{\pi \omega \sigma \theta_2} & \frac{\omega}{\varpi \sigma} & \frac{1}{\sigma} \end{pmatrix}. \quad (29)$$

The dominant eigen value in  $G = FV^{-1}$  is the value of  $R_0 = \rho(FV^{-1}) = \rho G$ . It is the spectral radius of matrix G.

$$FV^{-1} = \begin{pmatrix} \frac{(\lambda\omega\pi_1 + \lambda\pi_1\omega)\varepsilon\tilde{\eta}_0}{\pi\sigma\omega\theta_1} + \frac{\lambda\pi_1\tilde{\eta}_0}{\pi\omega\theta_1} & \frac{(\lambda\omega\pi_2 + \lambda\pi_2\omega)\varepsilon\tilde{\eta}_0}{\pi\sigma\omega\theta_2} + \frac{\lambda\pi_2\tilde{\eta}_0}{\pi\omega\theta_2} & \frac{\omega\varepsilon\tilde{\eta}_0}{\sigma\omega} + \frac{\tilde{\eta}_0}{\omega} & \frac{\varepsilon\tilde{\eta}_0}{\sigma} \\ \frac{(\lambda\omega\pi_1 + \lambda\pi_1\omega)\varepsilon\tilde{\eta}_1}{\pi\sigma\omega\theta_1} + \frac{\lambda\pi_1\tilde{\eta}_1}{\pi\omega\theta_1} & \frac{(\lambda\omega\pi_2 + \lambda\pi_2\omega)\varepsilon\tilde{\eta}_1}{\pi\sigma\omega\theta_2} + \frac{\lambda\pi_2\tilde{\eta}_1}{\pi\omega\theta_2} & \frac{\omega\varepsilon\tilde{\eta}_1}{\sigma\omega} + \frac{\tilde{\eta}_1}{\omega} & \frac{\varepsilon\tilde{\eta}_1}{\sigma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigen values of the next generation matrix G are determined using Wolfram Mathematica as

$$\left\{ 0, 0, 0, \frac{\lambda(\omega\pi_2\varepsilon\tilde{\eta}_1\theta_1 + \omega\pi_2\varepsilon\tilde{\eta}_1\theta_1 + \sigma\pi_2\tilde{\eta}_1\theta_1 + \omega\pi_1\varepsilon\tilde{\eta}_0\theta_2 + \omega\pi_1\varepsilon\tilde{\eta}_0\theta_2 + \sigma\pi_1\tilde{\eta}_0\theta_2)}{\omega\pi\theta_1\theta_2} \right\}, \quad (30)$$

The dominant eigen value and hence the value of  $R_0$  is

$$R_0 = \left\{ \frac{\lambda\pi_2\tilde{\eta}_1}{\pi\omega\theta_2} + \frac{\varepsilon\lambda\pi_2\tilde{\eta}_1}{\pi\sigma\theta_2} + \frac{\varepsilon\lambda\omega\pi_2\tilde{\eta}_1}{\pi\omega\sigma\theta_2} + \frac{\lambda\pi_1\tilde{\eta}_0}{\pi\omega\theta_1} + \frac{\varepsilon\lambda\pi_1\tilde{\eta}_0}{\pi\sigma\theta_1} + \frac{\varepsilon\lambda\omega\pi_1\tilde{\eta}_0}{\pi\omega\sigma\theta_1} \right\}. \quad (31)$$

#### 4.4.1 Biological interpretation of $R_0$ .

As shown in the studies (Ronoh et al., 2021), (Alemneh et al., 2019)  $R_0$  expresses the average expected secondary infections after an average complete; organic maize-normal larvae, organic maize-resistant larvae, insecticidal sprayed maize-normal larvae, insecticidal sprayed maize-resistant larvae interaction.  $R_0$  can then be expressed as the sum of interactions,

$$R_0 = R_0^1 + R_0^2 + R_0^3 + R_0^4 + R_0^5 + R_0^6, \quad (32)$$

Where,  $R_0^{In} = R_0^1 + R_0^2 + R_0^3$  and  $R_0^{Or} = R_0^4 + R_0^5 + R_0^6$ .

$$\text{With } R_0^1 = \frac{\lambda\pi_2\tilde{\eta}_1}{\pi\omega\theta_2}, \quad R_0^2 = \frac{\varepsilon\lambda\pi_2\tilde{\eta}_1}{\pi\sigma\theta_2}, \quad R_0^3 = \frac{\varepsilon\lambda\omega\pi_2\tilde{\eta}_1}{\pi\omega\sigma\theta_2}.$$

$$R_0^4 = \frac{\lambda\pi_1\tilde{\eta}_0}{\pi\omega\theta_1}, \quad R_0^5 = \frac{\varepsilon\lambda\pi_1\tilde{\eta}_0}{\pi\sigma\theta_1}, \quad R_0^6 = \frac{\varepsilon\lambda\omega\pi_1\tilde{\eta}_0}{\pi\omega\sigma\theta_1}.$$

$R_0^{In}$  represents the number of new infections on Insecticidal maize population  $N_O^M(t)$  arising from both the normal larvae  $L_N$  and the resistant larvae  $L_R$ ,

$$R_0^{In} = \frac{\lambda\pi_2\tilde{\eta}_1}{\pi\omega\theta_2} + \frac{\varepsilon\lambda\pi_2\tilde{\eta}_1}{\pi\sigma\theta_2} + \frac{\varepsilon\lambda\omega\pi_2\tilde{\eta}_1}{\pi\omega\sigma\theta_2}. \quad (33)$$

$R_0^{Or}$  represents the number of new infections on organic maize population  $N_0^M(t)$  arising from both the normal larvae  $L_N$  and the resistant larvae  $L_R$ ,

$$R_0^{Or} = \frac{\lambda\pi_1\tilde{\eta}_0}{\pi\omega\theta_1} + \frac{\varepsilon\lambda\pi_1\tilde{\eta}_0}{\pi\sigma\theta_1} + \frac{\varepsilon\lambda\omega\pi_1\tilde{\eta}_0}{\pi\omega\sigma\theta_1}. \quad (34)$$

#### 4.5 Stability of equilibrium points.

In this study, we determined global asymptotic stability for three model steady states using the Castillo-Chavez, the Perron eigenvector and the Lyapunov methods. The steady states are defined to have local asymptotic stability if the eigenvalues of the generated Jacobian matrix ( $J$ ) evaluated at each (E), have non-positive real parts. The steady states are asymptotically unstable if at least one of the eigenvalues has a positive real part. This is the linearity stability analysis theorem.

Let the model system of equations be denoted in vector form as

$$\frac{df}{dt} = f(y). \quad (35)$$

With,  $y = (S_0^M(t), I_0^M(t), S_I^M(t), I_I^M(t), L_N(t), L_R(t))$ .

We then write  $f(y)$  in matrix form as

$$f(y) = \begin{pmatrix} \rho - \beta_0 S_0^M - (1 - \theta_1)\pi S_0^M \\ \beta_0 S_0^M - \theta_1 \pi I_0^M \\ \rho - (1 - \theta_2)\pi S_I^M - \beta_I S_I^M \\ \beta_I S_I^M - \theta_2 \pi I_I^M \\ \lambda(e_N + \pi_1 I_0^M + \pi_2 I_I^M) - (\omega + \mu_L + \delta_N)L_N \\ ((1 - e_N) + \pi_1 I_0^M + \pi_2 I_I^M)\lambda + \omega L_N - (\mu_L + \delta_R)L_R \end{pmatrix}. \quad (36)$$

The matrix  $f(y)$  is used to evaluate Jacobian matrix of the model and thus the stability analysis for the various model stationary points.

### 4.5.1 Local stability analysis of Disease- free Equilibrium point ( $E_0$ )

#### Theorem 3

The disease-free steady state for the FAW larvae-maize interaction model described by the equations (3)- (8) is described to have local asymptotic stability if all the below conditions are satisfied:

1.  $a_1, a_2, a_3, a_4, a_5 > 0$ ,
2.  $a_1 a_2 - a_3 > 0$ ,
3.  $a_3(a_1 a_2 - a_3) - a_1(a_1 a_4 - a_5) > 0$ ,
4.  $a_3^2 a_4 - a_1^2 a_4^2 - a_1 a_5 a_2^2 > 0$ .

Otherwise,  $E_0$  is unstable.

#### Proof

Evaluating the Jacobian matrix at DFEP to get,

$$(J_f)_{E_0} = \begin{pmatrix} -(1 - \theta_1)\pi & 0 & 0 & 0 & -\tilde{\eta}_0 & -\varepsilon\tilde{\eta}_0 \\ 0 & -\theta_1\pi & 0 & 0 & \tilde{\eta}_0 & \varepsilon\tilde{\eta}_0 \\ 0 & 0 & -(1 - \theta_2)\pi & 0 & -\tilde{\eta}_I & -\varepsilon\tilde{\eta}_I \\ 0 & 0 & 0 & -\theta_2\pi & \tilde{\eta}_I & \varepsilon\tilde{\eta}_I \\ 0 & \lambda\pi_1 & 0 & \lambda\pi_2 & -\varpi & 0 \\ 0 & \lambda\pi_1 & 0 & \lambda\pi_2 & \omega & -\sigma \end{pmatrix}. \quad (37)$$

The first eigenvalue of the above Jacobian matrix is  $\lambda_1 = -(1 - \theta_1)\pi$  which is less than zero and hence stable. Applying the Routh-Hurwitz stability criterion as shown by (Clark, 1992), the roots to the characteristic equation (38) below have negative real parts iff the coefficients  $a_i$  are non-negative and matrices  $J > 0$  for  $i = 1, 2, 3, 4, 5$ .

$$J_{f_{E_0^*}} = \begin{pmatrix} -\theta_1\pi - \lambda & 0 & 0 & \tilde{\eta}_0 & \varepsilon\tilde{\eta}_0 \\ 0 & -(1 - \theta_2)\pi - \lambda & 0 & -\tilde{\eta}_I & -\varepsilon\tilde{\eta}_I \\ 0 & 0 & -\theta_2\pi - \lambda & \tilde{\eta}_I & \varepsilon\tilde{\eta}_I \\ \lambda\pi_1 & 0 & \lambda\pi_2 & -\varpi - \lambda & 0 \\ \lambda\pi_1 & 0 & \lambda\pi_2 & \omega & -\sigma - \lambda \end{pmatrix}.$$

Using Mathematica software, the characteristic polynomial of the above matrix is evaluated as

$$a_0\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0. \quad (38)$$

Where  $\lambda_i = 1(1)5$  are the eigenvalues.

By the Routh-Hurwitz criterion for stability analysis, the model system is proved to be locally asymptotically stable at disease-free steady state if and only if  $a_1, a_2, a_3, a_4, a_5 > 0$ ,  $a_1a_2 - a_3 > 0$ ,  $a_3(a_1a_2 - a_3) - a_1(a_1a_4 - a_5) > 0$  and  $a_3^2a_4 - a_1^2a_4^2 - a_1a_5a_2^2 > 0$ . As shown in the Routh-Hurwitz Table 3 below.

Hence all the eigenvalues are negative and thus DFE is locally asymptotically stable.

**Table 3:** Routh-Hurwitz stability criterion

$\lambda^5$	$a_0$	$a_2$	$a_4$
$\lambda^4$	$a_1$	$a_3$	$a_5$
$\lambda^3$	$\frac{a_4a_3 - a_5a_2}{a_4}$	$\frac{a_4a_1 - a_5a_0}{a_4}$	0
$\lambda^2$	$\frac{a_1a_2a_3 - a_3^2 - a_1^2a_4 - a_1a_5}{a_1a_2 - a_3}$	$\frac{a_5a_1a_2 - a_5a_3}{a_2a_1 - a_3}$	0
$\lambda^1$	$\frac{a_1a_2(a_3a_4 - a_2a_5) + a_5^2 + a_2a_3a_5 - (a_1^2a_4 - a_3^2)a_4}{a_1a_2 - a_3}$	0	0
$\lambda^0$	0	0	0

#### 4.5.2 Global stability analysis of Disease-free Equilibrium point ( $E_0$ )

Let define  $f(x)$  as,

$$f(x) = (F - V)x - F(x) + V(x), \text{ Where } x = (I_O^M, I_I^M, L_N, L_R). \quad (39)$$

Also, let define the value  $x' = (F - V)x - f(x)$  with the values of  $F$  and  $V$  as defined previously in section 4.4 in equations (27) and (28) respectively.  $\omega^T \geq 0$  is defined as the

perron eigenvector or the left eigenvector corresponding to the eigenvalue  $\rho(V^{-1}F) = \rho(FV^{-1}) = R_0$ .

**Theorem 4**

Let  $F$ ,  $V$ , and  $f(x)$  be defined as shown above. If  $f(x) \geq 0$  in  $\Omega \in R_+^6$  the model system,  $F \geq 0, V^{-1} \geq 0$ , and  $R_0 \leq 1$  then we have  $L = \omega^T V^{-1} x$  as the Lyapunov function for the model system of equations as shown in the study done by (Koutsourelakis, 2009).

**Proof**

We start by getting the derivative of  $L$  along the solutions to the model equations (3) – (8).

$$\begin{aligned} L' &= \omega^T V^{-1} x' = \omega^T V^{-1} (F - V)x - \omega^T V^{-1} f(x), \text{ since } x' = (F - V)x - f(x), \\ &= (R_0 - 1)\omega^T V^{-1} f(x). \end{aligned} \tag{40}$$

And since as shown above, we have that  $\omega^T \geq 0, V^{-1} \geq 0$ , and  $f(x) \geq 0$  in  $\Omega \in R_+^6$ , then the last term is negative. If  $R_0 \leq 1$ , then  $L' \leq 1$  in  $\Omega$  and thus  $L$  is taken to be the Lyapunov function for the model system. We determine

$$\omega^T = [\omega_1 \quad \omega_2 \quad \omega_3 \quad \omega_4]^T = \omega^T V^{-1} F = R_0 \omega^T = [0 \quad 0 \quad 0 \quad 1]^T. \tag{41}$$

Therefore,

$$V^{-1} = \begin{pmatrix} \frac{1}{\theta_1 \pi} & 0 & 0 & 0 \\ 0 & \frac{1}{\theta_2 \pi} & 0 & 0 \\ \frac{\lambda \pi_1}{\varpi \pi \theta_1} & \frac{\lambda \pi_2}{\varpi \pi \theta_2} & \frac{1}{\varpi} & 0 \\ \frac{\lambda \omega \pi_1 + \lambda \pi_1 \varpi}{\varpi \pi \sigma \theta_1} & \frac{\omega \lambda \pi_2 + \lambda \pi_2 \varpi}{\varpi \sigma \pi \theta_2} & \frac{\omega}{\varpi \sigma} & \frac{1}{\sigma} \end{pmatrix}. \tag{42}$$

as generated in section 4.4



$$\omega^T V^{-1} = \begin{bmatrix} \frac{\lambda\omega\pi_I + \lambda\pi_I\varpi}{\varpi\pi\sigma\theta_I} \\ \frac{\omega\lambda\pi_2 + \lambda\pi_2\varpi}{\varpi\sigma\pi\theta_2} \\ \frac{\omega}{\varpi\sigma} \\ \frac{1}{\sigma} \end{bmatrix}. \quad (43)$$

Thus,

$$L = \omega^T V^{-1} x = \left( \frac{\lambda\omega\pi_I + \lambda\pi_I\varpi}{\varpi\pi\sigma\theta_I} \right) I_0^M + \left[ \frac{\omega\lambda\pi_2 + \lambda\pi_2\varpi}{\varpi\sigma\pi\theta_2} \right] I_I^M + \left[ \frac{\omega}{\varpi\sigma} \right] L_N + \left[ \frac{1}{\sigma} \right] L_R. \quad (44)$$

is the Lyapunov function for the model system (3) – (8).

By Perron-Frobenius, we let

$$V^{-1}F, f(x) \geq 0 \text{ with } f(x_0) = 0, F \geq 0, V^{-1} \geq 0 \quad (45)$$

to be irreducible and positive in  $\Omega \in R_+^6$ , it follows then that  $\omega^T > 0$ . Hence, (Arsie & Ebenbauer, 2009) by Lasalle's Invariant principle,

$$L' = 0 \text{ shows that } \omega^T x = 0 \text{ and } x = 0. \quad (46)$$

Thus, our disease-free steady state ( $E_0$ ) is Globally asymptotically stable.

We also prove that the disease-free steady state has global asymptotic stability using the Castillo- Chavez method as shown below:

We apply the method established by (Castillo-chavez, 2001). We start first by rewriting the system of model equations in the form:

$$\frac{dX}{dt} = F(X, Z), \quad (47)$$

$$\frac{dZ}{dt} = G(X, Z), G(X, 0 = 0). \quad (48)$$

With  $X = (S_0^M, S_I^M) \in R_+^2$  representing the uninfected classes,  $Z = (I_0^M, I_I^M, L_N, \text{ and } L_R) \in R_+^4$  representing the infected and the infectious classes.  $E_0 =$

$(X^*, 0)$  denotes the disease-free equilibrium point of the system  $E_0 = (S_O^{M0}, I_O^{M0}, S_I^{M0}, I_I^{M0}, L_N^0, L_R^0) = (\frac{\rho}{(1-\theta_1)\pi}, 0, \frac{\rho}{(1-\theta_2)\pi}, 0, 0, 0)$ . (Equation 23)

According to the Castillo-Chavez stability theorem if the following conditions are satisfied in the points given above, then the global asymptotic stability of  $E_0$  is guaranteed.

The conditions include:

1.  $\frac{dX}{dt} = F(X, 0), X^*$  is Globally asymptotically stable. (49)

2.  $\frac{dZ}{dt} = D_Z G(X, 0)Z - \tilde{G}(X, Z), \tilde{G}(X, Z) \geq 0, \forall (X, Z) \in \Omega \in R_+^6$ . (50)

### Theorem 5

*The Disease-free Equilibrium point ( $E_0$ ) is globally asymptotically stable.*

### Proof

We start by dividing the model into subsystems

$$X = (S_O^M, S_I^M), \text{ and } Z = (I_O^M, I_I^M, L_N, L_R). \quad (51)$$

We then generate two vector-valued functions

$$F(X, Z) = \begin{pmatrix} \rho - \beta_O S_O^M - (1 - \theta_1)\pi S_O^M \\ \rho - (1 - \theta_2)\pi S_I^M - \beta_I S_I^M \end{pmatrix}, \quad (52)$$

and

$$G(X, Z) = \begin{pmatrix} \beta_O S_O^M - \theta_1 \pi I_O^M \\ \beta_I S_I^M - \theta_2 \pi I_I^M \\ \lambda(\pi_1 I_O^M + \pi_2 I_I^M) - (\omega + \mu_L + \delta_N)L_N \\ (\pi_1 I_O^M + \pi_2 I_I^M)\lambda + \omega L_N - (\mu_L + \delta_R)L_R \end{pmatrix}. \quad (53)$$

Evaluating for the reduced system from condition (1),  $\frac{dX}{dt} = F(X, 0)$  to get,

$$\frac{dS_O^M}{dt} = \rho - (1 - \theta_1)\pi S_O^M, \quad \frac{dS_I^M}{dt} = \rho - (1 - \theta_2)\pi S_I^M$$

We note that the system has dynamics of an asymptomatic system independent of initial conditions in  $\Omega$ .

We then compute  $G(X, Z) = D_Z G(X^*, 0)Z - \tilde{G}(X, Z)$  and prove that  $\tilde{G}(X, Z) \geq 0$ .

Let  $A = D_Z G(X^*, 0)$  a Jacobian matrix of  $\tilde{G}(X, Z)$  taken in  $Z = (I_0^M, I_I^M, L_N, L_R)$  and evaluated at  $(X^*, 0)$ . It is also defined as an M matrix since all the non-diagonal elements that are non-negative.

$$A = \begin{pmatrix} \theta_1 \pi & 0 & 0 & 0 \\ 0 & \theta_2 \pi & 0 & 0 \\ \lambda \pi_1 & \lambda \pi_2 & -\varpi & 0 \\ \lambda \pi_1 & \lambda \pi_2 & \omega & -\sigma \end{pmatrix}. \quad (54)$$

Evaluating the value of  $AZ$

$$AZ = \begin{pmatrix} \theta_1 \pi I_0^M \\ \theta_2 \pi I_I^M \\ \lambda \pi_1 I_0^M + \lambda \pi_2 I_I^M - \varpi L_N \\ \lambda \pi_1 I_0^M + \lambda \pi_2 I_I^M + \omega L_N - \sigma L_R \end{pmatrix}. \quad (55)$$

With the equation  $\tilde{G}(X, Z) = AZ - G(X, Z) \geq 0$ , then the value of

$$\tilde{G}(X, Z) = \begin{pmatrix} \beta_0 S_0^M \\ \beta_I S_I^M \\ 0 \\ 0 \end{pmatrix}. \quad (56)$$

Since  $\tilde{G}(X, Z) \geq 0 \quad \forall (X, Z) \in \Omega \in R_+^6$  then the DFE is proved to be globally asymptotically stable.

### 4.5.3 Local stability analysis of Control free Equilibrium point

#### Theorem 6

The control-free equilibrium ( $E_C$ ) point is Locally asymptotically stable.

#### Proof

Local stability analysis of the control free steady state is achieved by first generating the Jacobian matrix  $J_{f(E_C)}$  evaluated at control free steady-state evaluated in section 4.4,

$$J_{f(E_C)} = \begin{pmatrix} -(1 - \theta_1)\pi - a_0 & a_1 & a_1 & a_1 & a_2 & a_3 \\ a_0 & -a_1 - \theta_1\pi & -a_1 & a_1 & a_2 & a_3 \\ 0 & 0 & -(1 - \theta_2)\pi - a_4 & 0 & 0 & 0 \\ 0 & 0 & a_5 & -\theta_2\pi & 0 & 0 \\ 0 & \lambda\pi_1 & 0 & \lambda\pi_2 & -\varpi & 0 \\ 0 & \lambda\pi_1 & 0 & \lambda\pi_2 & \omega & -\sigma \end{pmatrix} \quad (57)$$

$$\text{Where } a_0 = \frac{(L_N^c + \epsilon L_R^c)\eta_0}{S_O^{Mc} + I_O^{Mc}} - \frac{S_O^{Mc}(L_N^c + \epsilon L_R^c)\eta_0}{(S_O^{Mc} + I_O^{Mc})^2}, \quad a_1 = \frac{S_O^{Mc}(L_N^c + \epsilon L_R^c)\eta_0}{(S_O^{Mc} + I_O^{Mc})^2}, \quad a_2 = \frac{\eta_0 S_O^{Mc}}{S_O^{Mc} + I_O^{Mc}},$$

$$a_3 = \frac{\epsilon \eta_0 S_O^{Mc}}{S_O^{Mc} + I_O^{Mc}}, \quad a_4 = \frac{(L_N^c + \epsilon L_R^c)\eta_0}{S_O^{Mc} + I_O^{Mc}}, \quad \text{and } a_5 = \frac{(L_N^c + \epsilon L_R^c)\eta_1}{S_O^{Mc} + I_O^{Mc}}, \quad \text{with } L_N^c, L_R^c, S_O^{Mc}, I_O^{Mc} \text{ as shown}$$

in section 4.4.

Using Wolfram Mathematica, the eigen values are found to be,

$$\begin{aligned} \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = & (\pi\varpi\sigma a_1)^4 \\ & + ((\pi + \varpi + \sigma + a_0 + a_1) - \pi\lambda\sigma a_2\pi_1 - 2\lambda\sigma a_0 a_2\pi_1 - \pi\lambda\varpi a_3\pi_1 \\ & - \pi\lambda\omega a_3\pi_1 - 2\lambda\varpi a_0 a_3\pi_1 - 2\lambda\omega a_0 a_3\pi_1 + \pi^2\varpi\sigma\theta_1 + \pi\varpi\sigma a_0\theta_1 \\ & - \pi\varpi\sigma a_1\theta_1 + \pi\lambda\sigma a_2\pi_1\theta_1 + \pi\lambda\varpi a_3\pi_1\theta_1 + \pi\lambda\omega a_3\pi_1\theta_1 - \pi^2\varpi\sigma\theta_1^2)^3 \\ & + (\pi\varpi + \pi\sigma + \varpi\sigma + \varpi a_0 + \sigma a_0 + \pi a_1 + \varpi a_1 + \sigma a_1 - \lambda a_2\pi_1 \\ & - \lambda a_3\pi_1 + \pi^2\theta_1 + \pi a_0\theta_1 - \pi a_1\theta_1 - \pi^2\theta_1^2)^2 \\ & + (\pi\varpi\sigma + \varpi\sigma a_0 + \pi\varpi a_1 + \pi\sigma a_1 + \varpi\sigma a_1 - \pi\lambda a_2\pi_1 - \lambda\sigma a_2\pi_1 \\ & - 2\lambda a_0 a_2\pi_1 - \pi\lambda a_3\pi_1 - \lambda\varpi a_3\pi_1 - \lambda\omega a_3\pi_1 - 2\lambda a_0 a_3\pi_1 + \pi^2\varpi\theta_1 \\ & + \pi^2\sigma\theta_1 + \pi\varpi a_0\theta_1 + \pi\sigma a_0\theta_1 - \pi\varpi a_1\theta_1 - \pi\sigma a_1\theta_1 + \pi\lambda a_2\pi_1\theta_1 \\ & + \pi\lambda a_3\pi_1\theta_1 - \pi^2\varpi\theta_1^2 - \pi^2\sigma\theta_1^2), \end{aligned}$$

$$\gamma_5 = -\pi\theta_2, \text{ and } \gamma_6 = -\pi - a_4 + \pi\theta_2. \quad (58)$$

Through back substitution all the eigen values are negative and thus the control free equilibrium point is evaluated to be stable.

#### 4.5.4 Global stability analysis of Control free equilibrium ( $E_C$ ) point.

If a Lyapunov function to a linearized nonlinear system is obtained and exists, then that shows that the model system is asymptotically stable (Al-Sheikh, 2012).

#### Theorem 7

*The Control free equilibrium point is globally asymptotically stable: (i) If the control free equilibrium is feasible and (ii) If the equilibrium point is a locally asymptotically stable solution.*

#### Proof

We consider a Lyapunov method for stability analysis an approach adopted by (Maini & Korobeinikov, 2004). We start by constructing a Lyapunov function

$$L = \sum b_i (q_i - q_i^c \ln q_i). \quad (59)$$

With  $b_i$  representing a constant selected such that  $b_i > 0$ ,  $q_i$  representing the  $i^{th}$  compartments classes, and  $q_i^c$  representing the control free equilibrium point of the  $i^{th}$  compartmental classes.

Expanding the Lyapunov function and substituting the compartments

$$L = b_1(S_0^M - S_0^{Mc} \ln S_0^M) + b_2(I_0^M - I_0^{Mc} \ln I_0^M) + b_3(S_I^M - S_I^{Mc} \ln S_I^M) + b_4(I_I^M - I_I^{Mc} \ln I_I^M) + b_5(L_N - L_N^c \ln L_N) + b_6(L_R - L_R^c \ln L_R). \quad (60)$$

We evaluate the derivative of the above equation (60) with respect to time to get,

$$\begin{aligned} \frac{dL}{dt} = & b_1 \left(1 - \frac{S_0^{Mc}}{S_0^M}\right) \frac{dS_0^M}{dt} + b_2 \left(1 - \frac{I_0^{Mc}}{I_0^M}\right) \frac{dI_0^M}{dt} + b_3 \left(1 - \frac{S_I^{Mc}}{S_I^M}\right) \frac{dS_I^M}{dt} + b_4 \left(1 - \frac{I_I^{Mc}}{I_I^M}\right) \frac{dI_I^M}{dt} + \\ & b_5 \left(1 - \frac{L_N^c}{L_N}\right) \frac{dL_N}{dt} + b_6 \left(1 - \frac{L_R^c}{L_R}\right) \frac{dL_R}{dt}. \end{aligned} \quad (61)$$

Where:  $S_0^M(t) = S_0^{Mc}(t), I_0^M(t) = I_0^{Mc}(t), S_I^M(t) = S_I^{Mc}(t), I_I^M(t) = I_I^{Mc}(t), L_N(t) = L_N^c(t), L_R(t) = L_R^c(t)$ .

Substituting the values of  $\frac{dS_0^M}{dt}, \frac{dI_0^M}{dt}, \frac{dS_I^M}{dt}, \frac{dI_I^M}{dt}, \frac{dL_N}{dt}, \frac{dL_R}{dt}$  with the model values to get

$$\begin{aligned}
\frac{dL}{dt} = & b_1 \left(1 - \frac{S_0^{Mc}}{S_0^M}\right) [\rho - \beta_O S_0^M - (1 - \theta_1)\pi S_0^M] + b_2 \left(1 - \frac{I_0^{Mc}}{I_0^M}\right) [\beta_O S_0^M - \theta_1 \pi I_0^M] \\
& + b_3 \left(1 - \frac{S_I^{Mc}}{S_I^M}\right) [\rho - (1 - \theta_2)\pi S_I^M - \beta_I S_I^M] \\
& + b_4 \left(1 - \frac{I_I^{Mc}}{I_I^M}\right) [\beta_I S_I^M - \theta_2 \pi I_I^M] \\
& + b_5 \left(1 - \frac{L_N^c}{L_N}\right) [\lambda(e_N + \pi_1 I_0^M + \pi_2 I_I^M) - (\omega + \mu_L + \delta_N)L_N] \\
& + b_6 \left(1 - \frac{L_R^c}{L_R}\right) [((1 - e_N) + \pi_1 I_0^M + \pi_2 I_I^M)\lambda + \omega L_N \\
& - (\mu_L + \delta_R)L_R].
\end{aligned} \tag{62}$$

As shown in section 4.2 the model equations are positively invariant, hence

$$\frac{dL}{dt} \leq 0 \quad \forall S_0^M(t), I_0^M(t), S_I^M(t), I_I^M(t), L_N(t), L_R(t) > 0. \tag{63}$$

also

$$\begin{aligned}
\frac{dL}{dt} = 0 \quad \text{if } S_0^M(t) = S_0^{Mc}(t), I_0^M(t) = I_0^{Mc}(t), S_I^M(t) = S_I^{Mc}(t), I_I^M(t) = \\
I_I^{Mc}(t), L_N(t) = L_N^c(t), L_R(t) = L_R^c(t).
\end{aligned} \tag{64}$$

Thus, in the set  $\{S_0^{Mc}(t), I_0^{Mc}(t), S_I^{Mc}(t), I_I^{Mc}(t), L_N^c(t), L_R^c(t) \in \Omega \in R_+^6\}$  the largest invariant set where  $\frac{dL}{dt} = 0$  is the singleton  $(E_C)$ , that is, the control free equilibrium point.

By Lasalle's invariant principle (Arsie & Ebenbauer, 2009),  $(E_C)$  is globally asymptotically stable in the set  $\Omega$  if  $R_0 \leq 1$

Otherwise, is unstable.

#### 4.5.5 Global stability analysis of endemic equilibrium ( $E_*$ ) point.

##### Theorem 8

For the Endemic equilibrium point to be globally stable, then (i) the endemic equilibrium point must be feasible, and (ii) the endemic equilibrium point must be locally stable.

##### Proof

We first assume that the Endemic equilibrium point is locally asymptotically stable since linearization method evaluated at the equilibrium point proves to be mathematically complicated.

Applying the Lyapunov method as used by (Maini & Korobeinikov, 2004) and constructing the appropriate Lyapunov function as

$$L = \sum a_i (x_i - x_i^* \ln x_i). \quad (65)$$

Where  $a_i$  represents a constant selected such that  $b_i > 0$ ,  $x_i$  represent the  $i^{th}$  compartments classes, and  $x_i^*$  represents the disease endemic equilibrium point( $E_*$ ) of the  $i^{th}$  compartment classes.

Expanding the Lyapunov function and substituting the compartments

$$L = a_1(S_0^M - S_0^{M*} \ln S_0^M) + a_2(I_0^M - I_0^{M*} \ln I_0^M) + a_3(S_I^M - S_I^{M*} \ln S_I^M) + a_4(I_I^M - I_I^{M*} \ln I_I^M) + a_5(L_N - L_N^* \ln L_N) + a_6(L_R - L_R^* \ln L_R). \quad (66)$$

Differentiating the above equation (66) with respect to time to get,

$$\begin{aligned} \frac{dL}{dt} = & a_1 \left(1 - \frac{S_0^{M*}}{S_0^M}\right) \frac{dS_0^M}{dt} + a_2 \left(1 - \frac{I_0^{M*}}{I_0^M}\right) \frac{dI_0^M}{dt} + a_3 \left(1 - \frac{S_I^{M*}}{S_I^M}\right) \frac{dS_I^M}{dt} + a_4 \left(1 - \frac{I_I^{M*}}{I_I^M}\right) \frac{dI_I^M}{dt} + \\ & a_5 \left(1 - \frac{L_N^*}{L_N}\right) \frac{dL_N}{dt} + a_6 \left(1 - \frac{L_R^*}{L_R}\right) \frac{dL_R}{dt}. \end{aligned} \quad (67)$$

Which is equivalent to

$$\begin{aligned} \frac{dL}{dt} = & a_1 \left(1 - \frac{S_0^{M*}}{S_0^M}\right) [\rho - \beta_O S_0^M - (1 - \theta_1)\pi S_0^M] + a_2 \left(1 - \frac{I_0^{M*}}{I_0^M}\right) [\beta_O S_0^M - \theta_1 \pi I_0^M] + \\ & a_3 \left(1 - \frac{S_I^{M*}}{S_I^M}\right) [\rho - (1 - \theta_2)\pi S_I^M - \beta_I S_I^M] + a_4 \left(1 - \frac{I_I^{M*}}{I_I^M}\right) [\beta_I S_I^M - \theta_2 \pi I_I^M] + a_5 \left(1 - \frac{L_N^*}{L_N}\right) \\ & [\lambda(e_N + \pi_1 I_0^M + \pi_2 I_I^M) - (\omega + \mu_L + \delta_N)L_N] + a_6 \left(1 - \frac{L_R^*}{L_R}\right) [(1 - e_N) + \pi_1 I_0^M + \\ & \pi_2 I_I^M]\lambda + \omega L_N - (\mu_L + \delta_R)L_R]. \end{aligned}$$

Where:  $S_0^M(t) = S_0^{M*}(t), I_0^M(t) = I_0^{M*}(t), S_I^M(t) = S_I^{M*}(t), I_I^M(t) = I_I^{M*}(t), L_N(t) = L_N^*(t), L_R(t) = L_R^*(t)$ .

As in section (4.2), the model equations are positively invariant, hence

$$\frac{dL}{dt} \leq 0 \quad \forall S_0^M(t), I_0^M(t), S_I^M(t), I_I^M(t), L_N(t), L_R(t) > 0. \quad (68)$$

and

$$\frac{dL}{dt} = 0 \quad \text{if } S_0^M(t) = S_0^{M*}(t), I_0^M(t) = I_0^{M*}(t), S_I^M(t) = S_I^{M*}(t), I_I^M(t) = I_I^{M*}(t), L_N(t) = L_N^*(t), L_R(t) = L_R^*(t). \quad (69)$$

Thus, in the set  $\{S_0^{M*}(t), I_0^{M*}(t), S_I^{M*}(t), I_I^{M*}(t), L_N^*(t), L_R^*(t) \in \Omega \in R_+^6\}$  the largest invariant set such that  $\frac{dL}{dt} = 0$  is the Singleton  $(E_*)$ , that is, the disease endemic equilibrium point. By Lasalle's invariant principle (Arsie & Ebenbauer, 2009),  $(E_*)$  is globally asymptotically stable in the set  $\Omega$  if  $R_0 \leq 1$  the interior of  $\Omega$ , otherwise unstable.

#### 4.6 Sensitivity Analysis of $R_0$ .

Sensitivity analysis of  $R_0$  helps identify the key parameters that significantly affect the FAW larvae-maize interaction model. This helps determine the key parameters to consider in the control strategies against the FAW larvae infestation into the maize population by managing the basic reproduction number and the force of infection. We follow a sensitivity method as conducted by (Alemneh et al., 2019) and (Garcia et al., 2019).



**Definition 4.6.0** The normalized forward sensitivity index of a variable,  $R_0$ , depending differentially on parameter  $P$  is defined by an equation

$$\alpha_{P}^{R_0} = \frac{\partial R_0}{\partial P} \cdot \frac{P}{R_0} \quad (70)$$

Where  $R_0$  represents the Basic reproduction number

$P$  represents all the main parameters.

In our study, we have the value of  $R_0$  given as,

$$R_0 = \frac{\lambda(\varepsilon\varpi + \sigma + \varepsilon\omega)(\pi_2\theta_1\tilde{\eta}_I + \pi_1\theta_2\tilde{\eta}_0)}{\pi\varpi\sigma\theta_1\theta_2} .$$

The sensitivity index of  $R_0$  to  $\lambda$  is

$$\alpha_{\lambda}^{R_0} = \frac{\partial R_0}{\partial \lambda} \cdot \frac{\lambda}{R_0} , \quad (71)$$

$$= \frac{(\varepsilon\varpi + \sigma + \varepsilon\omega)(\pi_2\theta_1\tilde{\eta}_I + \pi_1\theta_2\tilde{\eta}_0)}{\pi\varpi\sigma\theta_1\theta_2} \cdot \frac{\lambda}{\frac{\lambda(\varepsilon\varpi + \sigma + \varepsilon\omega)(\pi_2\theta_1\tilde{\eta}_I + \pi_1\theta_2\tilde{\eta}_0)}{\pi\varpi\sigma\theta_1\theta_2}} = 1 \quad (72)$$

A similar procedure is used to calculate the sensitivity indices for the other parameters around the basic reproduction number; the results are shown in the table below.

**Table 4:** Sensitivity Indices

Parameter	Sensitivity index
$\lambda$	1
$\sigma$	-0.5
$\varpi$	-0.5
$\varepsilon$	0.5
$\omega$	0.2

$\pi$	-0.5
$\pi_1$	0.5
$\pi_2$	0.4
$\theta_1$	-0.5
$\theta_2$	-0.4

#### 4.6.1 Interpreting the sensitivity Indices

Sensitivity indices for the basic reproduction number  $R_0$  presented in the table above show positive and negative values for the parameter values. A positive sensitivity index denotes a direct linkage between the parameter and the basic reproduction number while a negative sensitivity index denotes an inverse linkage between the parameter and the basic reproduction number. From the table,  $(\lambda, \varepsilon, \omega, \pi_1, \text{ and } \pi_2)$  have positive sensitivity indices. This means that the parameter values have a great impact on spreading the disease among the maize population upon an increase in their parameter values. This is because the value of  $R_0$  will tend to increase by increasing the parameter values, further increasing the number of secondary infections in the susceptible maize population.

The parameter  $(\sigma, \varpi, \theta_1, \theta_2)$  have negative sensitivity indices. This means that the value of  $R_0$  decreases when their values are increased. This results in a decrease in the rate of infection, lowering the secondary infections in the susceptible maize population. The value of  $\lambda$  being equal to 1 means a unit increase in  $\lambda$  results in a unit increase in the value of  $R_0$  and vice versa.

### 4.7 Numerical analysis

#### 4.7.1 Parameter estimation.

Numerical analysis of the model is conducted using a Matlab inbuilt solver based on Runge-Kutta order 5 with parameter values cited in table 5 as obtained from published studies together with a few estimated values. The initial value states are used as  $S_0^M(0) = 1000, I_0^M(0) = 0, S_I^M(0) = 1000, I_I^M(0) = 0, L_N(0) = 100, L_R(0) = 10$  as shown

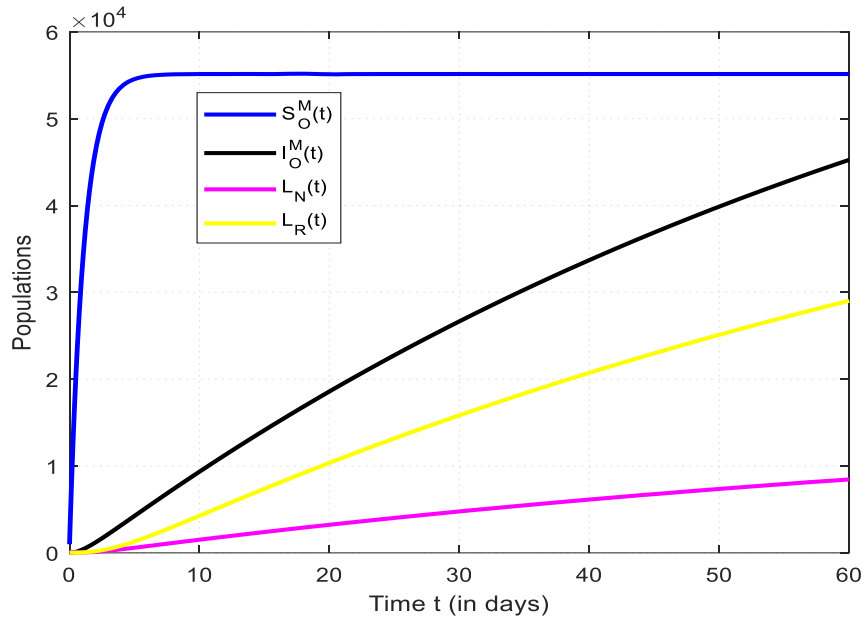
by similar studies on Fall Armyworm- Maize interaction (Daudi et al., 2021), (Alemneh et al., 2019) and (Jamieson, 2019). The simulations are conducted at a time range of between 0 to 60 days which is the vegetative stage of the maize population and the most interactive phase with the FAW population. The resulted simulation graphs are presented in the figures 2 to 11 below.

**Table 5:** Parameter values, ranges, and references.

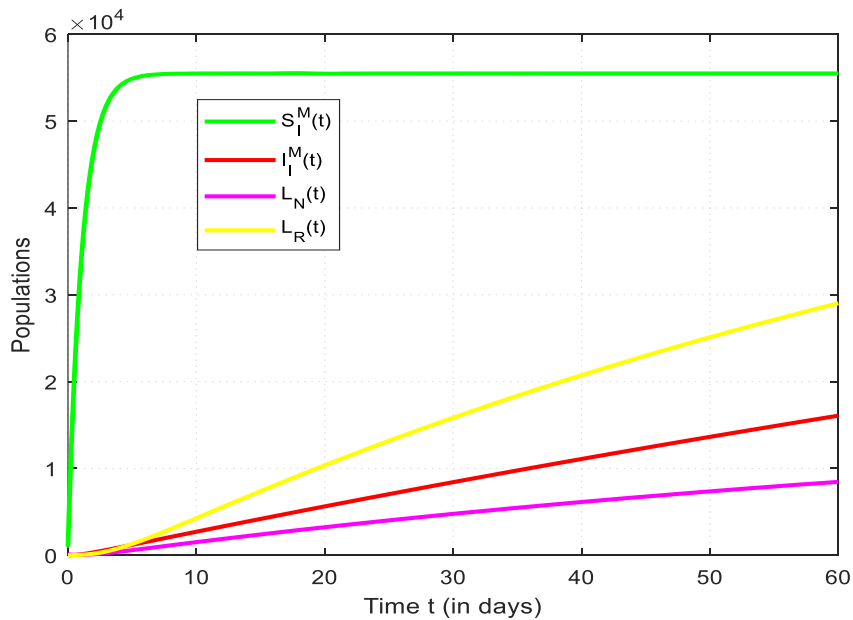
Parameter	Description	Parameter value	Source/Reference
$\theta_1$	The harvesting rate of organic maize population $N_O^M(t)$	0.015	(Alemneh et al., 2019)
$K$	Maximum plant carrying capacity of the two maize sections	1000 plants	(Alemneh et al., 2019)
$\theta_2$	The harvesting rate of insecticidal sprayed maize population $N_I^M(t)$	0.005	Estimated
$e_N$	The natural recruitment rate of larvae from the naturally occurring FAW population.	0.98	(De Groote et al., 2020)
$\omega$	The rate at which normal larvae progress into resistant larvae population	0.45	(Daudi et al., 2021)
$\rho$	The natural recruitment rate of the maize biomass into the maize population.	50Kg per plant	(Daudi et al., 2021)
$\mu_L$	Total population decrease rate of the larvae	0.077	Calculated
$\mu_2$	Progression rate into the pupa FAW life cycle.	0.071	(De Groote et al., 2020)
$\mu_1$	The natural death rate of the FAW larvae	0.0071	(De Groote et al., 2020)

$\delta_R$	The insecticidal induced death rates on the resistant larvae.	0.35	(Jamieson, 2019)
$\delta_N$	The insecticidal-induced death rates in the normal larvae	0.52	(Jamieson, 2019)
$\eta_0$	Infection factor	0.3922	Estimated
$\eta_1$	Infection factor	0.1087	Estimated
$\beta_0$	The rate of infection in the organic maize population	0.0202	Calculated
$\beta_1$	The rate of infection in the insecticidal maize population	0.0056	Calculated
$\lambda$	The survival rate of the larvae from the egg stage of the FAW population	0.75	(Daudi et al., 2021)
$\pi$	The lost maize biomass in the $N_M(t)$ class was due to a caterpillar attack.	0.9	(Jamieson, 2019)
$\pi_1$	The maize biomass from the $I_O^M(t)$ class contributing directly to the larvae increased natural recruitment rate.	0.2	(Jamieson, 2019)
$\pi_2$	The maize biomass from the $I_I^M(t)$ class contributing directly to the larvae increased natural recruitment rate.	0.18	Estimated

### 4.7.2 Simulation results.



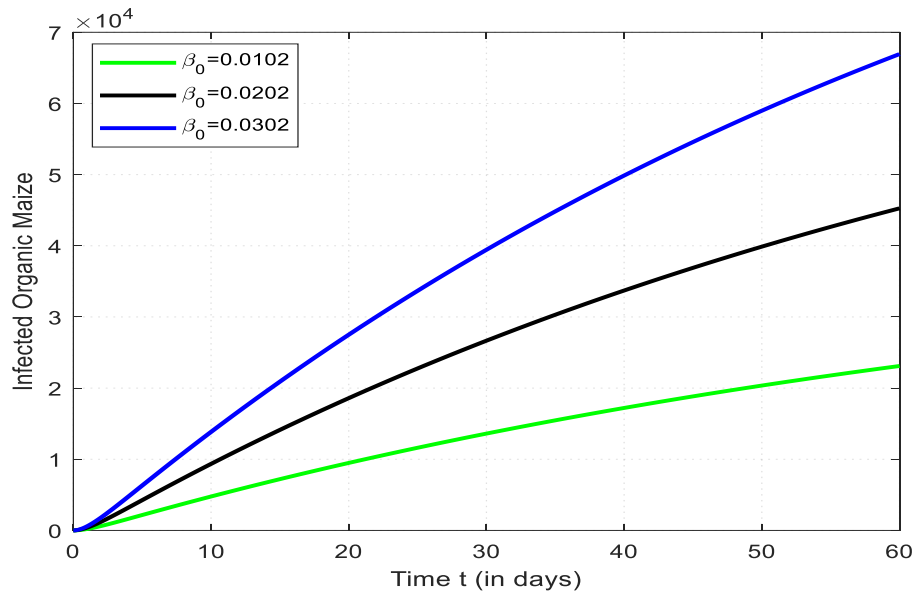
**Figure 2:** Population dynamics of organic maize interacting with the normal and resistant larvae.



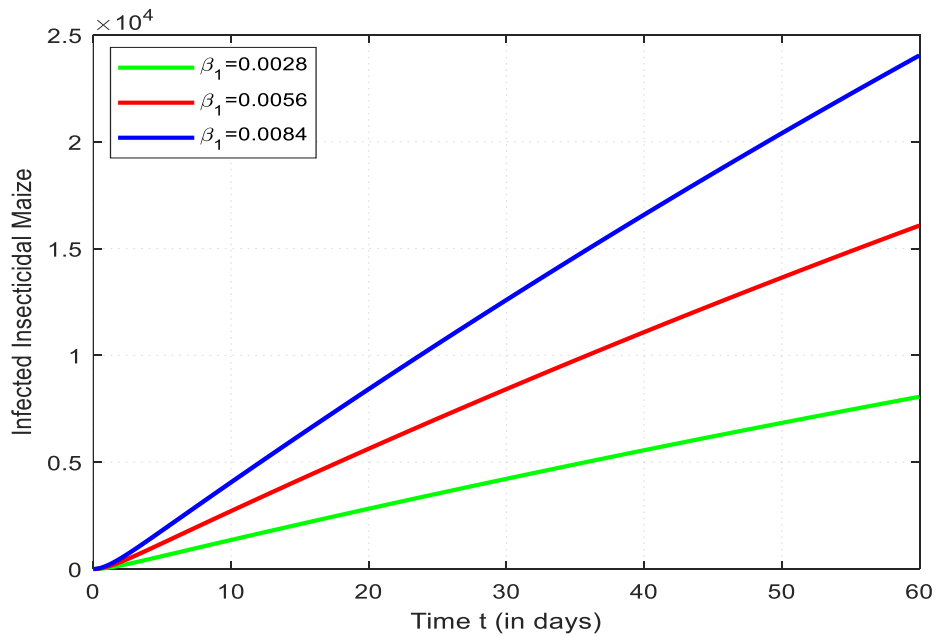
**Figure 3:** Population dynamics of insecticidal maize interacting with the normal and resistant larvae.

In figure 2 and figure 3 above, the susceptible maize populations  $\{S_O^M(t), S_I^M(t)\}$  increase exponentially with time until they reach the endemic equilibrium point. The curve for the

organic infected maize in (figure 2) is observed to be higher than the curve for the infected insecticidal maize population in (figure 3). This is the effect of uncontrolled FAW larvae-organic maize interactions which subsequently leads to a high force of infection ( $\beta_0$ ) resulting to a higher infected organic maize population.

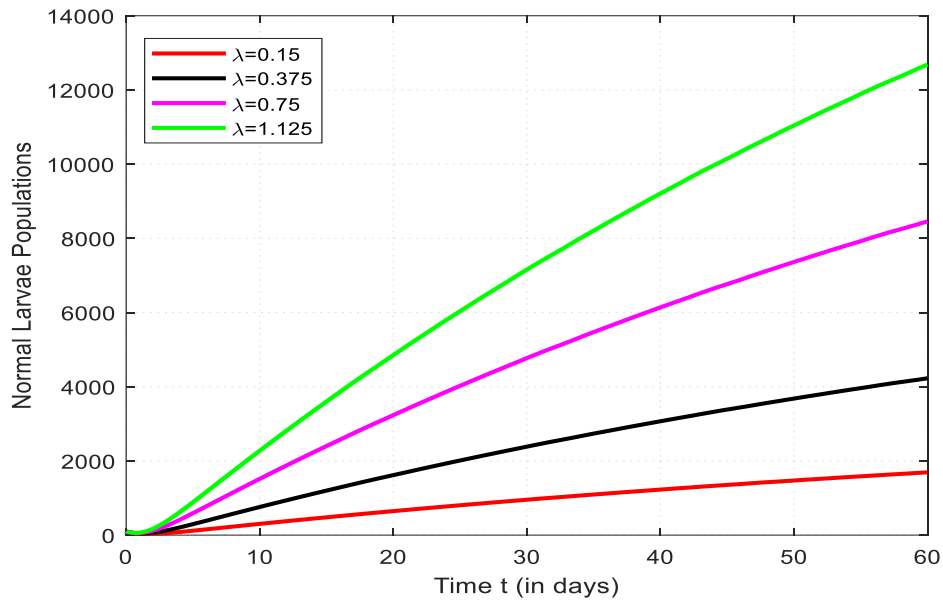


**Figure 4:** Population dynamics of infected organic maize at distinct values of  $\beta_0$ .

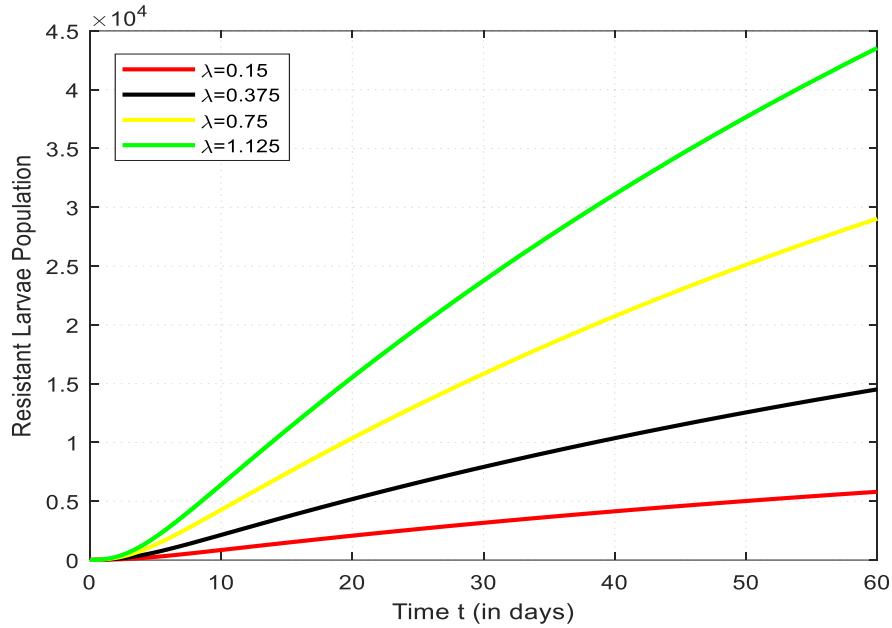


**Figure 5:** Population dynamics of infected insecticidal maize at distinct values of  $\beta_1$ .

In figure 4 above, when  $\beta_0$  is decreased by 50% the infected organic maize population decreases exponentially with time. This is due to the direct linkage between the force of infection and the basic reproduction number. At a higher value of  $\beta_0$ , the infected organic maize population increases exponentially. Similar observations are made in figure 5 however, the infected insecticidal maize population occurs in lower rates as compared to the organic section. This is due to the lower values of  $\beta_1$  as compared to  $\beta_0$ .



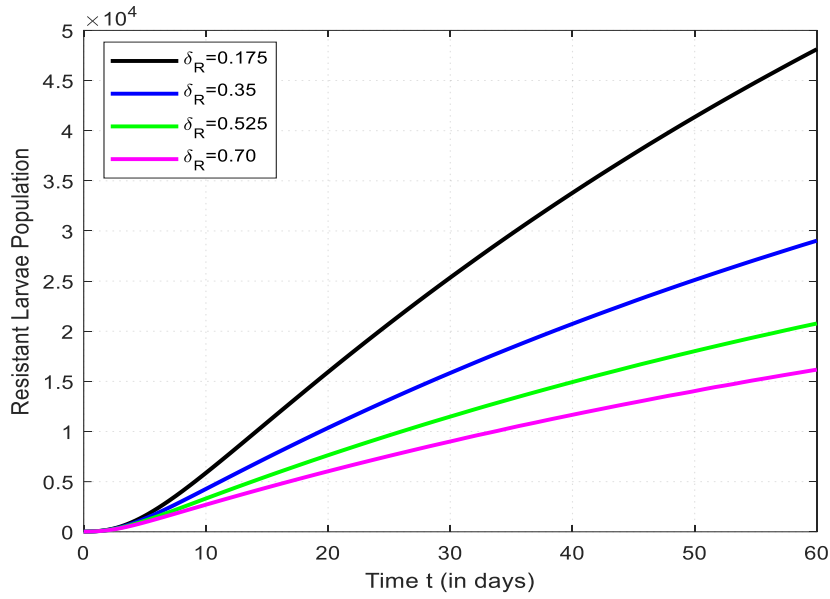
**Figure 6:** Population dynamics of normal larvae at distinct values of  $\lambda$ .



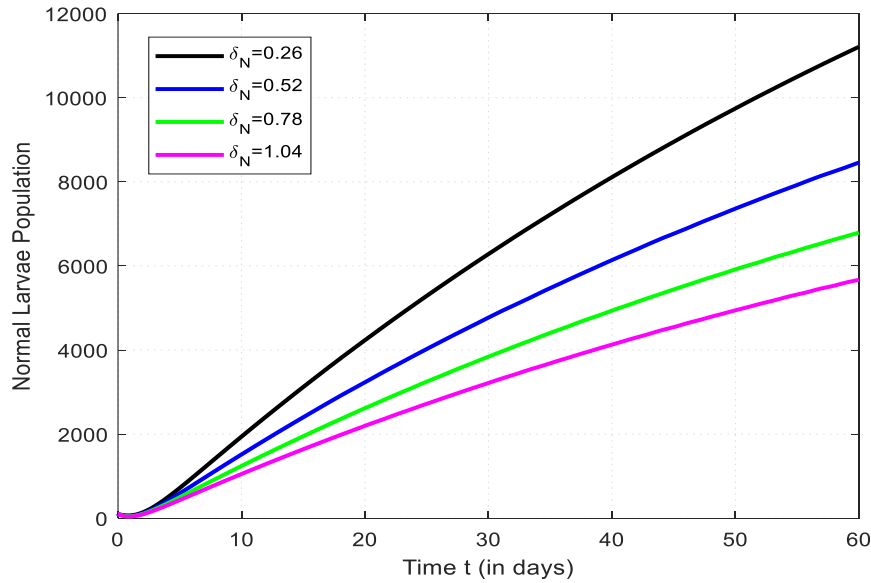
**Figure 7:** Population dynamics of resistant larvae at distinct values of  $\lambda$ .

According to figure 1 of the model flow chart,  $\lambda$  represents the survival rate for the FAW larvae. From figure 6 and figure 7, reducing the value of  $\lambda$  to lower values than the baseline value  $\lambda = 0.75$  subsequently reduces the population of the normal and the resistant larvae. Increasing the value of  $\lambda$  results to an increase in the larvae populations. This subsequently results to an increase in the maize-FAW interactions and thus higher infection rates in the maize populations. Thus, a direct effect on the force of infection in the maize population. To reduce the infection rates in the model, we need to lower the values of  $\lambda$ .





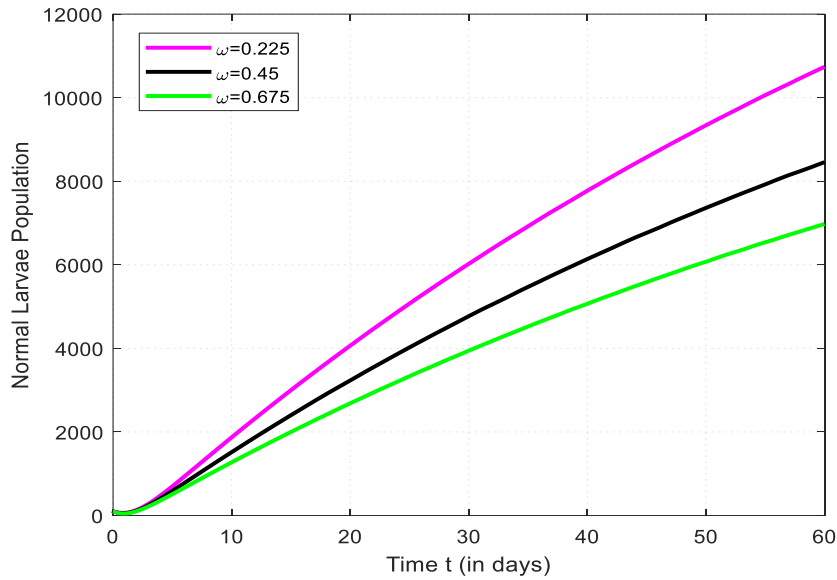
**Figure 8:** Population dynamics of resistant larvae at distinct values of  $\delta_R$ .



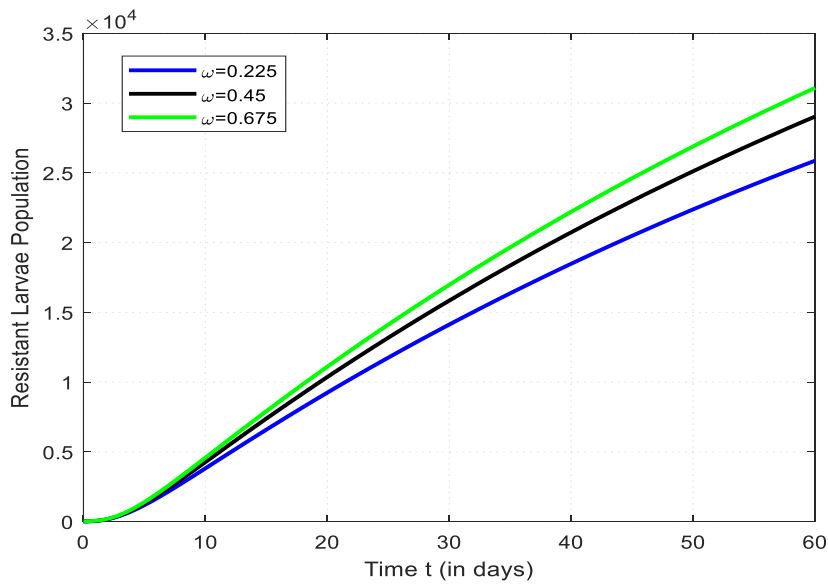
**Figure 9:** Population dynamics of normal larvae at distinct values of  $\delta_N$ .

$\delta_R$  and  $\delta_N$  denote the insecticidal-induced death rate on the resistant and normal larvae respectively. In figure 8 and figure 9, increasing the values of  $\delta_R$  and  $\delta_N$  reduces the population of the resistant and the normal larvae. This in turn reduces the infection rate leading to a lower basic reproduction number. We also observe that the resistant larvae

are large in numbers as compared to normal larvae which pose a greater risk in the insecticides control measures.



**Figure 10:** Population dynamics of normal larvae at distinct values of  $\omega$ .



**Figure 11:** Population dynamics of resistance larvae at distinct values of  $\omega$ .

From figure 10 and figure 11 above,  $\omega$ , represents the rate at which the normal larvae progress into resistant larvae after contact with insecticidal sprays. At higher values of  $\omega$  more normal larvae which are easy to eradicate from the model population by insecticides,

progress into resistant larvae. This subsequently reduces the normal larvae population and increases the resistant larvae population.

## CHAPTER FIVE

### DISCUSSION, CONCLUSION, AND RECOMMENDATION

#### 5.1 Discussion

This study developed and analyzed a deterministic eco-epidemiological model on maize-FAW interaction in presence of insecticides and resistance factors. The model was proved to be uniformly bounded and positively invariant. Three equilibrium points, that is, the disease/larvae free, control free, and endemic equilibrium points were established and evaluated to be locally and globally asymptotically stable at  $R_0 \leq 1$ . Further, an expression for the basic reproduction number  $R_0$  and its sensitivity analysis were conducted. The results showed that an increase on  $\omega, \lambda, \beta_0, \beta_1$  and a decrease on  $\delta_R, \delta_N$  greatly increased the FAW larvae interaction and population dynamics and hence the spread of the disease to the susceptible maize population. Through numerical simulation, graphical results of the FAW-maize interaction and population dynamics are presented by applying parameter values obtained from the literature and cited accordingly.

The analysis of  $R_0$  in host-pest (Maize-FAW) interaction model helps determine how effective the insecticides control measures against the FAW larvae are and how and when to effectively use the control measure to reduce the value of  $R_0$  to a value of less than one (Daudi et al., 2022). Increasing the forces of infection ( $\beta_0, \beta_1$ ) resulted to an increase in number of infected maize populations while reducing the infection forces resulted to a decrease in the number of infected maize populations in both the organic and insecticidal sections. This was attributed to the direct impact of the force of infection to the basic reproduction number. The insecticides control measures used to control FAW-maize interactions in the insecticidal maize section ensured a reduced contact rate and thus the number of resulting secondary infections at any time  $t$  were lower as compared to the organic uncontrolled FAW-maize interactions.

$\omega$  is a parameter value used in the model to represent a constant rate at which the normal larvae progress into resistant larvae after insecticides spraying. When insecticides are used to control the FAW larvae population, a few mutants in the population tolerate the chemical insecticides better while the normal population succumbs to them (Charaabi et

al., 2018). Resistance  $\omega$  increased the infection rates by increasing the FAW larvae survival rate  $\lambda$  and reducing the insecticidal efficacy by lowering the larvae insecticidal-induced death rates  $\delta_R$  and  $\delta_N$ . However, while resistance affects both the organic and the insecticidal controlled maize sections, the FAW-maize interactions and thus the infection rates are lower in the controlled insecticidal sections as compared to the organic uncontrolled section.

## 5.2 Conclusion and Recommendation

The developed and analyzed deterministic maize- FAW interaction model showed that the FAW survival rate  $\lambda$ , resistance formation  $\omega$ , and the insecticidal induced death rates  $\delta_R$  and  $\delta_N$  are very essential in controlling both the normal and the resistant FAW larvae. Control intervention aimed at reducing the infection rate in organic and insecticidal maize populations should aim at reducing these parameter factors. This is by using high-efficacy insecticides resulting in higher FAW larvae death rates ( $\delta_R, \delta_N$ ) thus reducing the FAW survival rate,  $\lambda$ . The sensitivity analysis of  $R_0$  showed that the FAW survival rate  $\lambda$  significantly affects the FAW- maize interactions. This informs both the organic and the inorganic farmers on the importance of using chemical control methods that are highly effective in reducing the FAW survival rate  $\lambda$ .

Also, various integrated FAW-maize management approaches could be adopted where several pest control methods are used together since no control method has been reported to work best in isolation. African countries should conduct proper civic education on pest control methods, FAW-maize interaction patterns, resistance formation in insecticides use to ensure environmental conservation and minimize pest resistance formation. This is due to the FAW unique characteristics of high migration, mutation, and reproduction which makes its control a bit expensive and difficult. However, the findings from this study are not exhaustive. In future studies, we will consider developing an optimal larvae survival control theory with resistance factors to achieve a profitable FAW larvae control strategy.

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