Cycle Index Formulas for D_n Acting on Ordered Pairs

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Abstract: ***The cycle index of dihedral group $D_n$acting on the set X of the vertices of a regular n-gon was studied by Harary and Palmer in 1973 [1]. Since then a number of researchers have studied the cycle indices dihedral group acting on different sets and the resulting formulas have found applications in enumeration of a number of items. Muthoka (2015) [2] studied the cycle index formula of the dihedral group acting on unordered pairs from the set $X = \{1, 2, \ldots, n\}$ –the n vertices of a regular n-gon. In this paper we study the cycle index formulas of D_n acting on ordered pairs from the set $X = \{1, 2, \ldots, n\}$. In each case the actions of the cyclic part and the reflection part are studied separately for both an even value of n and an odd value of n.***

Keywords: Cycle index, Cycle type, Monomial

1. Introduction

The concept of the cycle index was discovered by Polya [3] and he gave it its present name. He used the cycle index to count graphs and chemical compounds via the Polya's Enumeration Theorem. Other current cycle index formulas include the cycle index of the reduced ordered triples groups $S_n^{[3]}$ [4] which was further extended by Kamuti and Njuguna to cycle index of the reduced ordered r-group $S_n^{[r]}$ [5]. The Cycle Index of Internal Direct Product Groups was done in 2012 [6].

2. Definitions and Preliminary Results

Definition 1
A cycle index is a polynomial in several variables which is structured in such a way that information about how a group of permutations acts on a set can be simply read off from the coefficients and exponents.

Definition 2
A cycle type of a permutation is the data of how many cycles of each length are present in the cycle decomposition of the permutation.

Definition 3
A monomial is a product of powers of variables with nonnegative integer exponents possibly with repetitions.

Preliminary result 1
Let (G,X) be a finite permutation group and let $X^{[2]}$ denote the set of all ordered 2-element subsets of X. If $g$is a permutation in (G,X) we want to know the disjoint cycle structure of the permutation g' induced by $g$on $X^{[2]}$.

We shall briefly sketch the technique (we call it the reduced pair group action) for obtaining the disjoint cycle structure of g'.
Let $mon(g) = t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_n^{\alpha_n}$, our aim is to find $mon\ (g')$. To do this there are two separate contributions from g to the corresponding term of $mon(g')$ which we need to consider:
(i) from pairs of points both lying in a common cycle of g.
(ii) from pairs of points each from a different cycle of g.
It is convenient to divide the first contribution into:

(a) Those pairs from odd cycles,
(b) Those pairs from even cycles.

(i) (a) Let $\theta = (123 \ldots 2m+1)$ be an odd cycle in g, then the permutation θ' in $(G, X^{[2]})$ induced by θ is as follows:

$\{123 \ldots 2m+1\} \rightarrow$

- $\{1,2\} \rightarrow \{2,3\} \rightarrow \{3,4\} \rightarrow \ldots \rightarrow \{2m+1,1\}$
- $\{1,3\} \rightarrow \{2,4\} \rightarrow \{3,5\} \rightarrow \ldots \rightarrow \{2m,1\} \rightarrow \{2m+1,2\}$
- $\{1,4\} \rightarrow \{2,5\} \rightarrow \{3,6\} \rightarrow \ldots \rightarrow \{2m-1,1\} \rightarrow \{2m,2\} \rightarrow \{2m+1,3\}$
- . . . .
- $\{1,m\} \rightarrow \{2,m+1\} \rightarrow \ldots \rightarrow \{m+3,1\} \rightarrow \ldots \rightarrow \{2m+1,m-1\}$
- $\{1,m+1\} \rightarrow \{2,m+2,\} \rightarrow \ldots \rightarrow \{m+2,1\} \rightarrow \ldots \rightarrow \{2m+1,m\}$
- $\{2,1\} \rightarrow \{3,2\} \rightarrow \{4,3\} \rightarrow \ldots \rightarrow \{1,2m+1\}$
- $\{3,1\} \rightarrow \{4,2\} \rightarrow \{5,3\} \rightarrow \ldots \rightarrow \{1,2m\} \rightarrow \{2,2m+1\}$
- . . . .
- $\{m,1\} \rightarrow \{m+1,2\} \rightarrow \{m+2,3\} \rightarrow \ldots \rightarrow \{1,m+3\} \rightarrow \ldots \{m-1,2m+1\}$
- $\{m+1,1\} \rightarrow \{m+2,2\} \rightarrow \ldots \rightarrow \{1,m+2\} \rightarrow \ldots \rightarrow \{m,2m+1\}$

Hence $t_{2m+1} \longrightarrow b_{2m+1}^{2m}$

So if we have α_{2m+1} cycles of length $2m+1$ in g, the pairs of points lying in a common cycle contribute:

$$t_{2m+1}^{\alpha_{2m+1}} \longrightarrow b_{2m+1}^{2m(\alpha_{2m+1})} \quad (2.1)$$

for odd cycles.

(i) (b) If $\theta = (123 \ldots 2m)$, then we get θ' as follows:

$\{123 \ldots 2m\} \rightarrow$

- $\{1,2\} \rightarrow \{2,3\} \rightarrow \{3,4\} \rightarrow \ldots \rightarrow \{2m,1\}$
- $\{1,3\} \rightarrow \{2,4\} \rightarrow \{3,5\} \rightarrow \ldots \rightarrow \{2m-1,1\} \rightarrow \{2m,2\}$
- . . . .
- $\{1,m-1\} \rightarrow \{2,m\} \rightarrow \{3,m+1\} \rightarrow \ldots \rightarrow \{m+3,1\} \rightarrow \ldots \rightarrow \{2m,m-2\}$
- $\{1,m+1\} \rightarrow \{2,m+2,\} \rightarrow \ldots \{m,2m\} \rightarrow \{m+1,1\} \rightarrow \ldots \rightarrow \{2m,m\}$
- $\{2,1\} \rightarrow \{3,2\} \rightarrow \{4,3\} \rightarrow \ldots \rightarrow \{1,2m\}$
- $\{3,1\} \rightarrow \{4,2\} \rightarrow \{5,3\} \rightarrow \ldots \rightarrow \{1,2m\} \rightarrow \{2,2m\}$
- . . . .
- $\{m-1,1\} \rightarrow \{m,2\} \rightarrow \{m+2,3\} \rightarrow \ldots \rightarrow \{1,m+3\} \rightarrow \ldots \{m-2,2m\}$
- $\{m,1\} \rightarrow \{m+1,2\} \rightarrow \ldots \rightarrow \{1,m+2\} \rightarrow \ldots \rightarrow \{m-1,2m\}$

The first m cycles above start with pairs of the form $(1,k)$ where $k = 2,3,\ldots,m+1$ giving a total of m cycles. The last $m-1$ cycles start with pairs of the form $(k,1)$ where $k = 2,3,\ldots,m+1$ giving a total of $m-1$ cycles.

Hence $t_{2m} \longrightarrow b_m b_{2m}^{2m-1}$

So if we have α_{2m} cycles of length $2m$ in g, the pairs of points lying in common cycle contribute;

$$t_{2m}^{\alpha_{2m}} \longrightarrow b_{2m}^{(2m-1)\alpha_{2m}} \quad (2.2)$$

for even cycles.

(ii) Consider two distinct cycles of length a and b in (G,X). If x belongs to an a-cycle θ_a of g and y belongs to a b-cycle θ_b of g, then the least positive integer β for which $g^{\beta}x = x$ and $g^{\beta}y = y$ is $[a,b]$, (the lcm of a and b). So the element (x,y) belongs to an $[a,b]$-cycle of g.

The number of such $[a,b]$-cycles contributed by g on $\theta_a \times \theta_b$ to g' is the total number of pairs in $\theta_a \times \theta_b$ divided by $[a,b]$, the length of each cycle. We note that the total number of pairs in $\theta_a \times \theta_b$ is twice the product ab.

This number is therefore $\frac{2ab}{[a,b]} = 2(a,b)$ that is twice the gcd of a and b.

In particular if $a = b = l$, the contribution by g on $\theta_a \times \theta_b$ to g' is $2l$ cycles of length l.

Thus when $a \neq b$ we have;

$$t_a^{\alpha_a} t_b^{\alpha_b} \longrightarrow b_{[a,b]}^{2(ab)\alpha_a\alpha_b} \quad (2.3)$$

and when $a = b = l$

$$t_l^{\alpha_l} \longrightarrow b_l^{2l\binom{\alpha_l}{2}} \quad (2.4)$$

Preliminary result 2

The cycle index formulas of dihedral group D_n acting on the set X of the vertices of a regular n-gon are given by:

$$Z_{D_n,X}=\frac{1}{2n}\left[\sum_{d|n}\phi(d)\,t_d^{\frac{n}{d}}+\frac{n}{2}t_1^2t_2^{\frac{n-2}{2}}+\frac{n}{2}t_2^{\frac{n}{2}}\right] \qquad 2.5(a)$$

if n is even and

$$Z_{D_n,X}=\frac{1}{2n}\left[\sum_{d|n}\phi(d)\,t_d^{\frac{n}{d}}+nt_1t_2^{\frac{n-1}{2}}\right] \qquad 2.5(b)$$

if n is odd. Where ϕ is the Euler's phi formula.

The proof to these important results can be found in several books and articles (e.g [6],[7])

3. Derivation of Cycle index of D_n acting on ordered pairs

With the help of the formulae for the cycle index of dihedral group D_n acting on the set X of the n vertices of a regular n-gon given in 2.5(a) and 2.5(b), we now study the cycle index of D_n acting on ordered pairs of the set $X=\{1,2,\ldots,n\}$.

3.1 Case 1: if n is even

We first consider the cyclic part $Z_{C_n}=\frac{1}{n}\sum_{d|n}\phi(d)t_d^{\frac{n}{d}}$

Since n is even, then the divisors of n can either be even or odd. We first consider when the divisor of n is even where we have two cases:

If both elements of a pair come from a common cycle, then from (2.2) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{\frac{n}{d}(d-1)} \qquad (3.1.1)$$

If each of the elements in a pair comes from different cycles of length d then from (2.4) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{2d\binom{n/d}{2}} \qquad (3.1.2)$$

Combining (3.1.1) and (3.1.2) we have

$$b_d^{2d\binom{n/d}{2}+\frac{n}{d}(d-1)} = b_d^{n(n-1)} \qquad (3.1.3)$$

If d is odd and the two elements come from a common cycle then from (2.1) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{\frac{n}{d}(d-1)} \qquad (3.1.4)$$

If both of the elements come from different cycles of length d then from (2.4) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{2d\binom{n/d}{2}}=b_d^{\frac{n(n-d)}{d}} \qquad (3.1.5)$$

Combining (3.1.4) and (3.1.5) we have;

$$b_d^{\frac{n}{d}(d-1)+\frac{n(n-d)}{d}}=b_d^{\frac{n(n-1)}{d}} \qquad (3.1.6)$$

Therefore the cycle index formula of $C_n$acting on $X^{[2]}$ when n is even is given by;

$$\frac{1}{n}\left[\sum_{\substack{d|n\\2|d}}\phi(d)\left(b_d^{\frac{n(n-1)}{d}}\right)+\sum_{\substack{d|n\\2\nmid d}}\phi(d)\left(b_d^{\frac{n(n-1)}{d}}\right)\right] \qquad (3.1.7)$$

From 2.5(a) we note that the two different kinds of reflections (a reflection through two vertices and a reflection through two sides) induce different monomials while acting on the vertices of a regular n-gon whenever n is even. We now investigate the induced monomials when the reflections act on $X^{[2]}$.

(i) We first consider the part $t_1^2t_2^{\frac{n-2}{2}}$.

If the elements in a pair come from different cycles of different lengths then from 2.3 we have:

$$t_1^2t_2^{\frac{n-2}{2}} \longrightarrow b_2^{2(n-2)} \qquad (3.1.8)$$

If both of the elements in a pair come from different cycles of the same length then we have two cases.

(a) Each from a cycle of length one and hence from 2.4 we have;

$$t_1^2 \longrightarrow b_1^2 \qquad 3.1.9(a)$$

(b) Each from a cycle of length two and hence from 2.4 we have;

$$t_2^{\frac{n-2}{2}} \longrightarrow b_2^{4\binom{\frac{n-2}{2}}{2}} \qquad 3.1.9(b)$$

Finally if both come from a common cycle of length two we note that two is even and hence from 2.2 we have;

$$t_2^{\frac{n-2}{2}} \longrightarrow b_2^{\frac{n-2}{2}} \qquad (3.1.10)$$

Combining (3.1.8), 3.1.9(a), 3.1.9(b) and (3.1.10) we have:

$$b_1^2b_2^{\frac{n-2}{2}+4\binom{\frac{n-2}{2}}{2}+2(n-2)}=b_1^2b_2^{\frac{(n+1)(n-2)}{2}}$$

From 2.5(a) we have $\frac{n}{2}$ monomials of the form $t_1^2t_2^{\frac{n-2}{2}}$ and hence a total of $\frac{n}{2}$ monomials will be induced from $t_1^2t_2^{\frac{n-2}{2}}$ giving;

$$\frac{n}{2}b_1^2b_2^{\frac{(n+1)(n-2)}{2}} \qquad (3.1.11)$$

(ii) Next we consider the part $t_2^{\frac{n}{2}}$

If the elements in a pair come from a common cycle then from 2.2 we have;

$$t_2^{\frac{n}{2}} \longrightarrow b_2^{\frac{n}{2}} \qquad (3.1.12)$$

If the elements in a pair come from different cycles of length two, then from 2.4 we have;

$$t_2^{\frac{n}{2}} \longrightarrow b_2^{4\binom{\frac{n}{2}}{2}} = b_2^{\frac{n(n-2)}{2}} \qquad (3.1.13)$$

Combining (3.1.12) and (3.1.13) we have; $b_2^{\frac{n(n-2)}{2}+\frac{n}{2}}=b_2^{\frac{n(n-1)}{2}}$; but from 2.5(a) we have $\frac{n}{2}$ monomials of the form $t_2^{\frac{n}{2}}$ and hence $\frac{n}{2}$ monomials will be induced giving;

$$\frac{n}{2}b_2^{\frac{n(n-1)}{2}} \qquad (3.1.14)$$

Adding the reflection parts given in (3.11) and (3.14) we have;

$$\frac{n}{2} b_1^2 b_2^{\frac{(n+1)(n-2)}{2}} + \frac{n}{2} b_2^{\frac{n(n-1)}{2}} \qquad (3.1.15)$$

Combining 3.7 and 3.15 we have the cycle index formula as;

$$Z_{D_n,X^{[2]}} = \frac{1}{2n}\left[\sum_{\substack{d|n\\2|d}} \phi(d)\left(b_d^{\frac{n(n-1)}{d}}\right) + \sum_{\substack{d|n\\2\nmid d}} \phi(d)\left(b_d^{\frac{n(n-1)}{d}}\right) + \frac{n}{2} b_1^2 b_2^{\frac{(n+1)(n-2)}{2}} + \frac{n}{2} b_2^{\frac{n(n-1)}{2}}\right] \qquad (3.1.16)$$

3.2 Case 2: If n is odd.

We first consider the cyclic part $Z_{C_n} = \frac{1}{n}\sum_{d|n} \phi(d) t_d^{\frac{n}{d}}$

In this case d must be odd because it is a divisor of an odd number.

If the elements in the pair come from a common cycle, then from 2.1 we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{\frac{n}{d}(d-1)} \qquad (3.2.1)$$

If the elements come from different cycles of length d then from 2.4 we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{2d\binom{\frac{n}{d}}{2}} = b_d^{\frac{n(n-d)}{d}} \qquad (3.2.2)$$

Combining (3.2.1) and (3.2.2) we have

$$b_d^{\frac{n(n-d)}{d}+\frac{n}{d}(d-1)} = b_d^{\frac{n(n-1)}{d}} \qquad (3.2.3)$$

Therefore the cycle index of C_n acting on $X^{[2]}$ when n is odd is given by;

$$\frac{1}{n}\left[\sum_{d|n} \phi(d)\, b_d^{\frac{n(n-1)}{d}}\right] \qquad (3.2.4)$$

To study the monomials induced by the reflection symmetries we note that all the reflection symmetries of a regular n-gon with n odd have their lines of symmetry passing through a vertex and an edge.

We now consider the monomials induced by the part $t_1 t_2^{\frac{n-1}{2}}$.

If the two elements in a pair come from a common cycle then from 2.2 we have;

$$t_2^{\frac{n-1}{2}} \longrightarrow b_2^{\frac{n-1}{2}} \qquad (3.2.5)$$

If the two elements come from different cycles then we have two cases.

(a) One element from a cycle of length one and the other from a cycle of length two. Then from (2.3) we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow b_2^{n-1} \qquad 3.2.6(a)$$

(b) If each come from a different cycle of length two then from 2.4 we have;

$$t_2^{\frac{n-1}{2}} \longrightarrow b_2^{4\binom{\frac{n-1}{2}}{2}} = b_2^{\frac{n^2-4n+3}{2}} \qquad 3.2.6(b).$$

Combining (3.2.5), 3.2.6(a) and 3.2.6(b) we have;

$$b_2^{4\binom{\frac{n-1}{2}}{2}+n-1+\frac{n-1}{2}} = b_2^{\frac{n(n-1)}{2}}$$

But from 2.5(b) we have n monomials of the form $t_1 t_2^{\frac{n-1}{2}}$ and hence n monomials will be induced giving;

$$n\, b_2^{\frac{n(n-1)}{2}} \qquad (3.2.7)$$

Combining (3.2.4) and (3.2.7) we have the cycle index formula as;

$$Z_{D_n,X^{[2]}} = \frac{1}{2n}\left[\sum_{d|n} \phi(d)\, b_d^{\frac{n(n-1)}{d}} + n\, b_2^{\frac{n(n-1)}{2}}\right] \qquad (3.2.8)$$

4. Conclusion

The cycle index formulas of D_n acting on ordered pairs are given as;

$$Z_{D_n,X^{[2]}} = \frac{1}{2n}\left[\sum_{\substack{d|n\\2|d}} \phi(d)\left(b_d^{\frac{n(n-1)}{d}}\right) + \sum_{\substack{d|n\\2\nmid d}} \phi(d)\left(b_d^{\frac{n(n-1)}{d}}\right) + \frac{n}{2} b_1^2 b_2^{\frac{(n+1)(n-2)}{2}} + \frac{n}{2} b_2^{\frac{n(n-1)}{2}}\right] \quad \text{if n is even}$$

And

$$Z_{D_n,X^{[2]}} = \frac{1}{2n}\left[\sum_{d|n} \phi(d)\, b_d^{\frac{n(n-1)}{d}} + n\, b_2^{\frac{n(n-1)}{2}}\right] \quad \text{if n is odd}$$

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